

# An Introduction to Nmatrices: Part I

## Many-valued Logics

## What Do We Mean By “Logic”?

1. A formal language  $\mathcal{L}$ , based on which  $\mathcal{L}$ -formulas are constructed. *We denote by  $F_{\mathcal{L}}$  the set of well-formed formulas of  $\mathcal{L}$ .*
2. A consequence relation  $\vdash$  for  $\mathcal{L}$ .

A **consequence relation (cr)** for  $\mathcal{L}$  is a binary relation  $\vdash \subseteq 2^{F_{\mathcal{L}}} \times F_{\mathcal{L}}$ , having the following properties:

*strong reflexivity:* if  $\psi \in \Gamma$  then  $\Gamma \vdash \psi$ .

*monotonicity:* if  $\Gamma \vdash \psi$  and  $\Gamma \subseteq \Gamma'$ , then  $\Gamma' \vdash \psi$ .

*transitivity (cut):* if  $\Gamma \vdash \psi$  and  $\Gamma, \psi \vdash \varphi$  then  $\Gamma \vdash \varphi$ .

## Properties of Consequence Relations

- A  $\text{cr } \vdash$  for  $\mathcal{L}$  is **structural** if for every uniform  $\mathcal{L}$ -substitution  $\sigma$  and every  $\Gamma$  and  $\psi$ : if  $\Gamma \vdash \psi$  then  $\sigma[\Gamma] \vdash \sigma[\psi]$ .  
**Example:**  $p \wedge q \vdash p$  implies  $\psi \wedge \varphi \vdash \psi$  for every  $\varphi, \psi \in F_{\mathcal{L}}$ .
- A  $\text{cr } \vdash$  for  $\mathcal{L}$  is **consistent** if there exist formulas  $\varphi$  and  $\psi$ , such that  $\varphi \not\vdash \psi$ .
- A  $\text{cr } \vdash$  for  $\mathcal{L}$  is **finitary** if whenever  $\Gamma \vdash \psi$ , there exists some finite  $\Gamma' \subseteq \Gamma$ , such that  $\Gamma' \vdash \psi$ .
- A **propositional logic** is a pair  $\langle \mathcal{L}, \vdash \rangle$ , where  $\vdash$  is a **structural, consistent and finitary** cr for  $\mathcal{L}$ .

## Example of a Language

- The language  $\mathcal{L}_{cl}$ :
  - *Atomic formulas:*  $p_1, p_2, \dots$ ,
  - *Logical connectives:*  $\neg, \wedge, \vee, \supset$
  - *Parentheses:*  $'(, ')'$
- The set of well-formed formulas  $F_{cl} = F_{\mathcal{L}_{cl}}$ :
  - *For any atomic formula  $p$ ,  $p \in F_{cl}$ .*
  - *If  $A, B \in F_{cl}$ , then  $(\neg A), (A \wedge B), (A \vee B), (A \supset B) \in F_{cl}$ .*
- Examples: **It is not raining and the sun is shining.**  
**If it does not rain and the car starts, then we are going for a trip.**

$$(\neg \text{Rain} \wedge \text{Sun}) \quad ((\neg \text{Rain} \wedge \text{CarStarts}) \supset \text{Trip})$$

## Ways of Defining Logics

- Semantically:  $\Gamma \vdash_S \psi$  if every “model” of  $\Gamma$  is a “model” of  $\psi$  in the semantics  $S$ .
- Syntactically:  $\Gamma \vdash_D \psi$  if  $\psi$  has a proof from  $\Gamma$  in the deduction system  $D$ .

## Classical Truth-Tables

$\neg$		$\supset$	t	f	$\wedge$	t	f	$\vee$	t	f
t	f	t	t	f	t	t	f	t	t	t
f	t	f	t	t	f	f	f	f	t	f

A **classical valuation** is a function  $v : F_{cl} \rightarrow \{t, f\}$ , such that  $v[\diamond(\psi_1, \dots, \psi_n)] = \tilde{\diamond}[v[\psi_1], \dots, v[\psi_n]]$  for any connective  $\diamond \in \{\neg, \supset, \vee, \wedge\}$ .

$$\begin{aligned}
 v[\text{Rain}] &= \mathbf{f} & v[\text{CarStarts}] &= \mathbf{t} & v[\text{Trip}] &= \mathbf{f} \\
 v[\neg \text{Rain}] &= \mathbf{t} & v[\neg \text{Rain} \wedge \text{CarStarts}] &= \mathbf{t} \\
 v[(\neg \text{Rain} \wedge \text{CarStarts}) \supset \text{Trip}] &= \mathbf{f}
 \end{aligned}$$

## Semantic Way of Defining Classical Logic

- A classical valuation  $v$  is a **model** of an  $\mathcal{L}$ -formula  $\psi$  if  $v(\psi) = \mathbf{t}$ .  
 $v$  is a **model** of a theory  $\Gamma$  if  $v$  is a model of every  $\psi \in \Gamma$ .
- $\Gamma \vdash_{CPL} \psi$  if every classical model of  $\Gamma$  is a model of  $\psi$ .
- Example:

$$\{\text{CarStarts} \supset \text{Trip}, \neg \text{Trip}\} \vdash_{CPL} \neg \text{CarStarts}$$

## Hilbert-style Proof Systems

- A **Hilbert-style proof system** for  $\mathcal{L}$  consists of: (i) an (effective) set of  $\mathcal{L}$ -formulas called **axioms**, and (ii) an (effective) set of **inference rules**.
- A **proof** of  $\psi$  from  $\Gamma$  in a Hilbert-style system  $\mathbf{H}$  is a finite sequence of  $\mathcal{L}$ -formulas, where the last formula is  $\psi$ , and each formula is: (i) an axiom of  $\mathbf{H}$ , (ii) a member of  $\Gamma$ , or (iii) is obtained from previous formulas in the sequence by applying some inference rule of  $\mathbf{H}$ .  $\psi$  is a **theorem** of  $\mathbf{H}$  if  $\psi$  has a proof in  $\mathbf{H}$  from  $\emptyset$ .
- *We denote  $\Gamma \vdash_{\mathbf{H}} \psi$  if  $\psi$  has a proof from  $\Gamma$  in  $\mathbf{H}$ .  $\vdash_{\mathbf{H}}$  is a finitary cr for any Hilbert-style system  $\mathbf{H}$ .*
- A system  $\mathbf{H}$  for  $\mathcal{L}$  is **sound** for a logic  $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$  if  $\vdash_{\mathbf{H}} \subseteq \vdash$ .  $\mathbf{H}$  is **complete** for  $\mathbf{L}$  if  $\vdash \subseteq \vdash_{\mathbf{H}}$ .



HCL<sup>+</sup>

- **Axiom schemata:**

**I1**  $\varphi \supset (\psi \supset \varphi)$

**I2**  $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$

**I3**  $((\psi \supset \varphi) \supset \psi) \supset \psi$

**C1**  $\varphi \wedge \psi \supset \varphi$

**C2**  $\varphi \wedge \psi \supset \psi$

**C3**  $\varphi \supset (\psi \supset \varphi \wedge \psi)$

**D1**  $\varphi \supset \varphi \vee \psi$

**D2**  $\psi \supset \varphi \vee \psi$

**D3**  $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$

- **Inference Rule:**

$$\frac{\psi \quad \psi \supset \varphi}{\varphi} MP$$

**HCL**

Obtained by adding to  $\mathbf{HCL}^+$  (possibly without **D3**) the following axioms concerning negation:

$$\mathbf{N1} \quad (\psi \supset \varphi) \supset (\psi \supset \neg\varphi) \supset \neg\psi$$

and either of the following:

$$\mathbf{N2} \quad \neg\neg\varphi \supset \varphi$$

$$\mathbf{N3} \quad \neg\varphi \vee \varphi$$

Soundness and completeness theorem for CPL:

$$\Gamma \vdash_{HCL} \psi \Leftrightarrow \Gamma \vdash_{CPL} \psi$$

## Gentzen-style Proof Systems

- Hilbert-style systems operate on  $\mathcal{L}$ -formulas. Gentzen-style systems operate on *sequents*.
- A **sequent**: an expression of the form  $\Gamma \Rightarrow \Delta$ , where  $\Gamma, \Delta$  are *finite* sets of  $\mathcal{L}$ -formulas.
- A **standard** Gentzen-type system for  $\mathcal{L}$  consists of:
  1. **Standard axiom**:  $\psi \Rightarrow \psi$ .
  2. **Structural Weakening and Cut rules**:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{ (Weakening)} \quad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta} \text{ (Cut)}$$

3. **Logical introduction rules for the connectives of  $\mathcal{L}$ .**

## Proofs in Gentzen-style Systems

- A **proof** of a sequent  $\Gamma \Rightarrow \Delta$  from a set of sequents  $\Theta$  in  $G$  is a finite sequence of sequents, where the last sequent is  $\Gamma \Rightarrow \Delta$ , and each sequent is: (i) an axiom of  $G$ , (ii) a member of  $\Theta$ , or (iii) is obtained from previous sequents in the sequence by applying some rule of  $G$ .
- $\Gamma \Rightarrow \Delta$  is **provable** in  $G$  if it has a proof from the empty set of sequents in  $G$ .
- $\Gamma \vdash_G \psi$  if there is some finite  $\Gamma' \subseteq \Gamma$ , such that  $\Gamma' \Rightarrow \psi$  is provable in  $G$ .
- $\vdash_G$  is a finitary cr. If  $G$  is standard then  $\vdash_G$  is also structural.

## The System GCPL

( $\alpha$ )  $\psi \Rightarrow \psi$

$$(Weakening) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

$$(Cut) \quad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}$$

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \neg) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi}$$

$$(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \supset) \quad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$$

$$(\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$$

$$(\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

**Example: Proof of  $\vdash_G \Rightarrow \neg(\psi \wedge \varphi) \supset \neg\psi \vee \neg\varphi$**

$$\begin{array}{c}
 \frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \neg\psi, \varphi} \text{ (Weakening)} \quad \frac{\psi \Rightarrow \psi}{\psi \Rightarrow \neg\varphi, \psi} \text{ (Weakening)} \\
 \frac{\varphi \Rightarrow \neg\psi, \varphi}{\Rightarrow \neg\psi, \neg\varphi, \varphi} (\Rightarrow \neg) \quad \frac{\psi \Rightarrow \neg\varphi, \psi}{\Rightarrow \neg\psi, \neg\varphi, \psi} (\Rightarrow \neg) \\
 \frac{\Rightarrow \neg\psi, \neg\varphi, \varphi}{\Rightarrow \neg\psi \vee \neg\varphi, \psi} (\Rightarrow \vee) \quad \frac{\Rightarrow \neg\psi, \neg\varphi, \psi}{\Rightarrow \neg\psi \vee \neg\varphi, \varphi} (\Rightarrow \vee) \\
 \hline
 \frac{\Rightarrow \neg\psi \vee \neg\varphi, \psi \wedge \varphi}{\neg(\psi \wedge \varphi) \Rightarrow \neg\psi \vee \neg\varphi} (\neg \Rightarrow) \\
 \frac{\neg(\psi \wedge \varphi) \Rightarrow \neg\psi \vee \neg\varphi}{\Rightarrow \neg(\psi \wedge \varphi) \supset \neg\psi \vee \neg\varphi} (\Rightarrow \supset)
 \end{array}$$

## Completeness and Cut-elimination

- A Gentzen-style System **admits cut-elimination** if whenever  $\Gamma \Rightarrow \Delta$  is provable in  $G$ ,  $\Gamma \Rightarrow \Delta$  also has a cut-free proof in  $G$ .
- **Completeness Theorem for Classical Logic:**
  - $\Gamma \vdash_{GCPL} \psi$  iff  $\Gamma \vdash_{CPL} \psi$
  - GCPL admits cut-elimination
- **Important corollary - GCPL has the subformula property:**  
*If  $\Gamma \Rightarrow \Delta$  has a derivation in GCPL, then all the formulas in this derivation are subformulas of the formulas in  $\Gamma \Rightarrow \Delta$ .*

## Basic Principles of Classical Logic

- **Bivalence:**  
Every proposition is either true or false (there are exactly two truth-values).
- **Inconsistency Intolerance:**  
A proposition and its negation cannot be both true.
- **Truth-Functionality:**  
The truth-value of a complex proposition is uniquely defined by the truth-values of its constituents.



## Many-valued Logic - Motivation

- Sometimes incomplete information prevents us from telling if something is true or not.
- Łukasiewicz, “On Determinism”, 1970: *If statements about future events are already true or false, then the future is as much determined as the past and differs from the past only in so far as it has not yet come to pass.*
- The idea: reject Bivalence by adding a third truth-value I, to be read as “possible”.

## Three-valued Łukasiewicz Logic

$\supset$	f	I	t
f	t	t	t
I	I	t	t
t	f	I	t

$\neg$	
f	t
I	I
t	f

- A legal valuation  $v$  is a **Łuk-model** of a formula  $\psi$  if  $v(\psi) = \mathbf{t}$ .  $v$  is a **Łuk-model** of a theory  $\Gamma$  if  $v$  is a Łuk-model of every  $\psi \in \Gamma$ .
- $\Gamma \vdash_{\text{Łuk}} \psi$  if every Łuk-model of  $\Gamma$  is a Łuk-model of  $\psi$ .
- **Examples:**

$$\vdash_{\text{Łuk}} \mathbf{I1} \quad [\varphi \supset (\psi \supset \varphi)]$$

$$\not\vdash_{\text{Łuk}} \mathbf{I2} \quad [(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)]$$

## Kleene and McCarthy Logics

- Modelling parallel vs. sequential computation
- The third truth-value **I** - for “undefined”
- Negation is defined like in Łukasiewicz three-valued logic.
- The notion of a **model** of a formula and the associated **cr** are defined like in Łukasiewicz three-valued logic.

$\vee$	f	I	t
f	f	I	t
I	I	I	t
t	t	t	t

Kleene

$\vee$	f	I	t
f	f	I	t
I	I	I	I
t	t	t	t

McCarthy



- The third truth-value **I** - for “paradoxical”
- Implication is defined as follows:

$$a \widetilde{\supset} b = \begin{cases} \mathbf{t} & \text{if } a = \mathbf{f} \\ b & \text{if } a \in \{\mathbf{t}, \mathbf{I}\} \end{cases}$$

- Other connectives are defined like in Kleene’s logic.
- A legal valuation  $v$  is a  **$J_3$ -model** of a formula  $\psi$  if  $v(\psi) \in \{\mathbf{t}, \mathbf{I}\}$ .  
 $v$  is a  **$J_3$ -model** of a theory  $\Gamma$  if  $v$  is a  $J_3$ -model of every  $\psi \in \Gamma$ .
- $\Gamma \vdash_{J_3} \psi$  if every  $J_3$ -model of  $\Gamma$  is a  $J_3$ -model of  $\psi$ .

- **Examples:**

$$\vdash_{J_2} \mathbf{I2} \quad [(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)]$$

$$\not\vdash_{J_3} \mathbf{N1} \quad [(\psi \supset \varphi) \supset (\psi \supset \neg\varphi) \supset \neg\psi]$$

## Hilbert-type System for $J_3$

Obtained by adding to  $\mathbf{HCL}^+$  the following axioms for negation:

$$\mathbf{N3} \quad \neg\psi \vee \psi$$

$$\mathbf{NN} \quad \neg\neg\psi \equiv \psi$$

$$\mathbf{NC} \quad \neg(\psi \wedge \varphi) \equiv \neg\psi \vee \neg\varphi$$

$$\mathbf{ND} \quad \neg(\psi \vee \varphi) \equiv \neg\psi \wedge \neg\varphi$$

$$\mathbf{NI} \quad \neg(\psi \supset \varphi) \equiv \psi \wedge \neg\varphi$$

where  $\psi \equiv \varphi \stackrel{\text{def}}{=} (\psi \supset \varphi) \wedge (\varphi \supset \psi)$ .

## Gentzen-type System for $J_3$

Obtained by deleting  $(\neg \Rightarrow)$  from **GCPL** (From  $\Gamma \Rightarrow \Delta, \varphi$  infer  $\neg\varphi, \Gamma \Rightarrow \Delta$ ), and adding instead the following rules:

$$\frac{\Gamma, \psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \supset \varphi) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma \Rightarrow \neg(\psi \supset \varphi), \Delta}$$

$$\frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \varphi)}$$

$$\frac{\Gamma, \neg\psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \varphi) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \varphi)}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}$$

## General Semantic Method for Defining Logics

- A **denotational semantics** for a language  $\mathcal{L}$  is a pair  $S = \langle S, \models_S \rangle$ , where  $S$  is a non-empty set of “valuations” and  $\models_S: S \times F_{\mathcal{L}}$ . A valuation is usually some mapping from  $F_{\mathcal{L}}$  to some set of “truth values”.
- For  $v \in S$ ,  $v$  is a **S-model** of  $\psi$  if  $v \models_S \psi$ .  $v$  is an **S-model** of  $\Gamma$  if  $v \models_S \psi$  for every  $\psi \in \Gamma$ .
- $\Gamma \vdash_S \psi$  if every S-model of  $\Gamma$  is an S-model of  $\psi$ .
- For any denotational semantics  $S = \langle S, \models_S \rangle$  for  $\mathcal{L}$ ,  $\vdash_S$  is a cr.  
*However,  $\langle \mathcal{L}, \vdash_S \rangle$  may not be a logic.*

## Many-valued Matrices as Denotational Semantics

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$  is a **matrix** for a propositional language  $\mathcal{L}$  if:

- $\mathcal{V}$  is a nonempty set of truth-values,
- $\emptyset \neq \mathcal{D} \subset \mathcal{V}$  (the set of **designated** truth-values),
- for every  $n$ -ary connective  $\diamond$  of  $\mathcal{L}$ ,  $\mathcal{O}$  includes an operation  $\widetilde{\diamond} : \mathcal{V}^n \rightarrow \mathcal{V}$



## Examples

- In classical logic:  $\mathcal{V} = \{t, f\}$ , while  $\mathcal{D} = \{t\}$ .
- In the 3-valued logics of Łukasiewicz, Kleene, and McCarthy:  
 $\mathcal{V} = \{t, f, I\}$ , while  $\mathcal{D} = \{t\}$ .
- In  $J_3$ :  $\mathcal{V} = \{t, f, I\}$ , while  $\mathcal{D} = \{t, I\}$ .

## Valuations

- A **valuation**  $v$  in a matrix  $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$  is any function from the set of  $\mathcal{L}$ -formulas to  $\mathcal{V}$  such that:

$$v[\diamond(\psi_1, \dots, \psi_n)] = \tilde{\diamond}[v[\psi_1], \dots, v[\psi_n]]$$

- $v$  is a **model** of an  $\mathcal{L}$ -formula  $\psi$  in  $\mathcal{M}$ , denoted by  $v \models_{\mathcal{M}} \psi$ , if  $v(\psi) \in \mathcal{D}$ .
- $v$  is a **model** of a theory  $\Gamma$  in  $\mathcal{M}$ , denoted by  $v \models_{\mathcal{M}} \Gamma$ , if  $v$  is a model of every  $\psi \in \Gamma$ .

## Consequence Relation Induced by a Matrix

- $\Gamma \vdash_{\mathcal{M}} \psi$  if for every valuation  $v$  in  $\mathcal{M}$ :  $v \models_{\mathcal{M}} \Gamma$  implies  $v \models_{\mathcal{M}} \psi$ .
- Let  $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$  be some logic.  
     $\mathbf{L}$  is **sound** for a matrix  $\mathcal{M}$  if  $\vdash \subseteq \vdash_{\mathcal{M}}$ .  
     $\mathbf{L}$  is **complete** for  $\mathcal{M}$  if  $\vdash_{\mathcal{M}} \subseteq \vdash$ .  
     $\mathcal{M}$  is a **characteristic** matrix for  $\mathbf{L}$  if  $\mathbf{L}$  is sound and complete for  $\mathcal{M}$ .
- For any matrix  $\mathcal{M}$  for  $\mathcal{L}$ ,  $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$  is a propositional logic.
- *Converse direction: is every propositional logic induced by a matrix?*

## Logics Induced by Matrices and their Families

- Let  $C$  be a family of matrices. We say that  $\Gamma \vdash_C \psi$  if for every  $\mathcal{M} \in C$ :  $\Gamma \vdash_{\mathcal{M}} \psi$ .
- Every propositional logic is induced by some family of matrices.
- A set of  $\mathcal{L}$ -formulas (theory)  $\Gamma$  is  $\vdash$ -consistent if there exists some  $\mathcal{L}$ -formula  $\psi$  such that  $\Gamma \not\vdash \psi$ .
- A logic  $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$  is **uniform** if for every two theories  $\Gamma_1, \Gamma_2$  and an  $\mathcal{L}$ -formula  $\psi$ :  $\Gamma_1 \vdash \psi$  whenever  $\Gamma_1, \Gamma_2 \vdash \psi$  and  $\Gamma_2$  is a  $\vdash$ -consistent theory with no atoms in common with  $\Gamma_1 \cup \{\psi\}$ .
- Łos & Suszko: A finitary propositional logic has a characteristic matrix iff it is uniform.

## Dunn-Belnap's Logic for Inconsistent and Incomplete Information

A framework for information collecting and processing:

Processor

Source 1: A is true, C is false

Source 2: A is false

Source 3: B is true

## Dunn-Belnap's Logic: 4 truth-values

Truth-values which can be assigned to a formula  $A$  are subsets of  $\{0, 1\}$ :

- $\mathbf{t} = \{1\}$ :  $P$  has information that  $A$  is true, but no information that  $A$  is false.
- $\mathbf{f} = \{0\}$ :  $P$  has information that  $A$  is false, but no information that  $A$  is true.
- $\top = \{0, 1\}$ :  $P$  has both information that  $A$  is false and information that  $A$  is true.
- $\perp = \emptyset$ :  $P$  has no information on  $A$ .

Belnap's Logic: 4 truth-values

Processor

Source 1: A is true, C is false

Source 2: A is false

Source 3: B is true

$$v(A) = \top \quad v(B) = \mathbf{t} \quad v(C) = \mathbf{f} \quad v(D) = \perp$$

## Belnap's Logic: 4 truth-values

*Assumption 1:* The sources provide information on atomic formulas, but not necessarily all of them.

*Assumption 2:*  $P$  respects the classical truth-tables in the following sense:

1.  $P$  ascribes 1 (0) to  $\neg\varphi$  iff it ascribes 0 (1) to  $\varphi$
2.  $P$  ascribes 1 to  $\varphi \vee \psi$  iff it ascribes 1 to either  $\varphi$  or  $\psi$
3.  $P$  ascribes 0 to  $\varphi \vee \psi$  iff it ascribes 0 to both  $\varphi$  and  $\psi$
4.  $P$  ascribes 1 to  $\varphi \wedge \psi$  iff it ascribes 1 to both  $\varphi$  and  $\psi$
5.  $P$  ascribes 0 to  $\varphi \wedge \psi$  iff it ascribes 0 to either  $\varphi$  or  $\psi$

*“The processor ascribes  $x \in \{0, 1\}$  to  $\psi$ ”:*  $x$  is included in the subset of  $\{0, 1\}$  which is assigned by the processor to  $\psi$ .



## Dunn-Belnap's Logic

- $\mathcal{V} = \{\mathbf{t}, \mathbf{f}, \top, \perp\}$
- $\mathcal{D} = \{\mathbf{t}, \top\}$
- The truth-tables for the connectives:

$\vee$	$\perp$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\perp$	$\perp$	$\perp$	$\mathbf{t}$	$\mathbf{t}$
$\mathbf{f}$	$\perp$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$	$\top$

$\wedge$	$\perp$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\perp$	$\perp$	$\mathbf{f}$	$\perp$	$\mathbf{f}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{t}$	$\perp$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\top$	$\mathbf{f}$	$\mathbf{f}$	$\top$	$\top$

$\neg$	
$\perp$	$\perp$
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$

## Corresponding Gentzen-type System

Obtained by deleting from **GCPL** **both** of the negation rules (as well as the implication rules), and adding instead the following rules:

$$\frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \varphi)}$$

$$\frac{\Gamma, \neg\psi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \varphi) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \varphi)}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}$$

## Fuzzy Logic: The Sorites Paradox

- 1 grain of wheat does not make a heap.
- If 1 grain of wheat does not make a heap then 2 grains of wheat do not.
- If 2 grains of wheat do not make a heap then 3 grains do not.
- ...
- If 9,999 grains of wheat do not make a heap then 10,000 do not.

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**10,000 grains of wheat do not make a heap???**

## Fuzzy Logics (L. Zadeh)

- Fuzzy Logic is an extension of Classical Logic.
- The idea: use infinitely many truth-values in  $[0,1]$
- The interpretations of disjunction and conjunction are monotonic, and interpretations of all connectives agree with the classical ones on 0 and 1.

## Gödel and Łukasiewicz Fuzzy Logics

$$\mathcal{V} = [0, 1] \quad \mathcal{D} = \{1\} \quad \mathcal{O} = \{\tilde{\vee}, \tilde{\wedge}, \tilde{\supset}, \tilde{\neg}\}$$

$$a \tilde{\vee} b = \max(a, b) \quad a \tilde{\wedge} b = \min(a, b) \quad \tilde{\neg} a = a \tilde{\supset} 0$$

$$a \tilde{\supset} b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases} \quad a \tilde{\supset} b = \begin{cases} 1 & \text{if } a \leq b \\ 1 - a + b & \text{if } a > b \end{cases}$$

(Gödel) (Łukasiewicz)

## Models and cr in Fuzzy Logics

- A legal valuation  $v$  for a fuzzy logic  $\mathbf{L}$  is an **L-model** of a formula  $\psi$  if  $v(\psi) = 1$ .  $v$  is an **L-model** of a theory  $\Gamma$  if  $v$  is an **L-model** of any  $\psi \in \Gamma$ .
- $\Gamma \vdash_L \psi$  if every **L-model** of  $\Gamma$  is an **L-model** of  $\psi$ .
- **Examples:**
  - $\vdash_L \psi \supset (\varphi \supset \psi)$
  - $\not\vdash_L \varphi \vee \neg\varphi$
  - $\vdash_{\text{Łukasiewicz}} \psi \equiv \neg\neg\psi$ , but  $\not\vdash_{\text{Gödel}} \psi \equiv \neg\neg\psi$
  - $\vdash_{\text{Gödel}} \mathbf{I}_2$ , but  $\not\vdash_{\text{Łukasiewicz}} \mathbf{I}_2$
- Both Gödel and Łukasiewicz fuzzy logics are decidable.

## Axioms for Gödel fuzzy logic

$$\mathbf{I1} \quad \varphi \supset (\psi \supset \varphi)$$

$$\mathbf{I2} \quad (\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$$

$$\mathbf{C1} \quad \varphi \wedge \psi \supset \varphi$$

$$\mathbf{C2} \quad \varphi \wedge \psi \supset \psi$$

$$\mathbf{C3} \quad \varphi \supset (\psi \supset \varphi \wedge \psi)$$

$$\mathbf{D1} \quad \varphi \supset \varphi \vee \psi$$

$$\mathbf{D2} \quad \psi \supset \varphi \vee \psi$$

$$\mathbf{D3} \quad (\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$$

$$\mathbf{N1} \quad (\psi \supset \varphi) \supset (\psi \supset \neg \varphi) \supset \neg \psi$$

$$\mathbf{N4} \quad \neg \varphi \supset (\varphi \supset \psi)$$

$$\mathbf{PL} \quad (\varphi \supset \psi) \vee (\varphi \supset \psi)$$

## Gödel and Łukasiewicz Finite-valued Logics

For a given finite natural number  $n$ , the  $n$ -valued counterparts of the fuzzy logics of Gödel and Łukasiewicz are obtained by taking

$$\mathcal{V} = \{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$$

Then  $\mathcal{D}$  and the interpretations of the connectives are defined analogously as for the infinite-valued logics.