

Introduction to Nmatrices: Part II

Introducing Nmatrices

Lack of Modularity in Matrices

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

		\neg
t	f	
f	t	

		\vee
t	t	t
t	f	t
f	t	t
f	f	f

Lack of Modularity in Matrices

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \text{ reject}}{\Gamma \Rightarrow \Delta, \neg\psi} \text{LEM}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

\neg	
t	f
f	???

\vee		
t	t	t
t	f	t
f	t	t
f	f	f

Syntactic Underspecification \Rightarrow Non-determinism

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

	\neg
t	{f}
f	{t,f}

		\vee
t	t	{t}
t	f	{t}
f	t	{t}
f	f	{f}

Introducing Non-determinism

- *Types of non-deterministic phenomena:*
vagueness incompleteness
uncertainty imprecision
inconsistency
- *Non-deterministic phenomena are in conflict with the principle of truth-functionality.*
- Representation idea: non-deterministic evaluation (interpretation) of formulas.

An example:

\diamond	t	f
t	{t}	{t, f}
f	{t, f}	{f}

Linguistic Ambiguity

In many natural languages, “or” has both inclusive and exclusive meanings. For example, if a mathematician says:

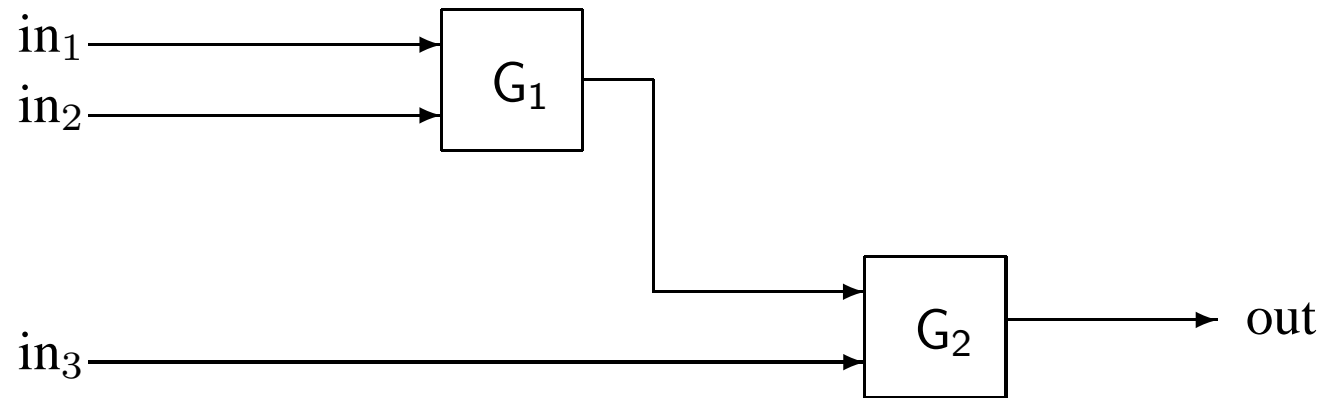
I shall either attack problem A or attack problem B

in many cases he may solve both problems, but in some situations he actually means “*but don’t expect me to solve both*”.

Capturing both meanings:

a	b	a OR b
t	t	{t, f}
t	f	{t}
f	t	{t}
f	f	{f}

Circuit Example

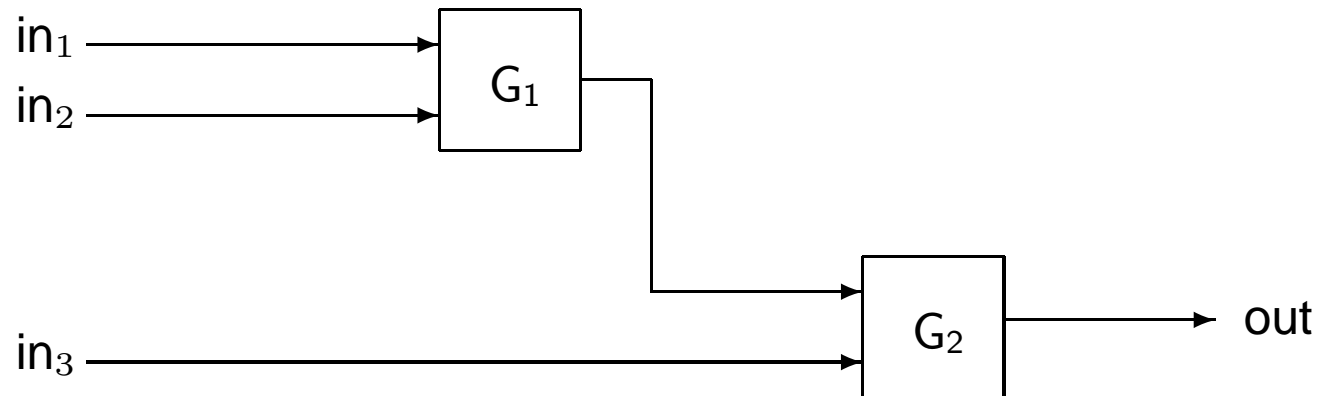


Here G_1 is an **OR** gate, and G_2 is an **XOR** gate

Unknown Boolean Functions

Suppose we know that each of the gates G_1 and G_2 is either an OR gate or an XOR gate, but we do not know which one exactly.

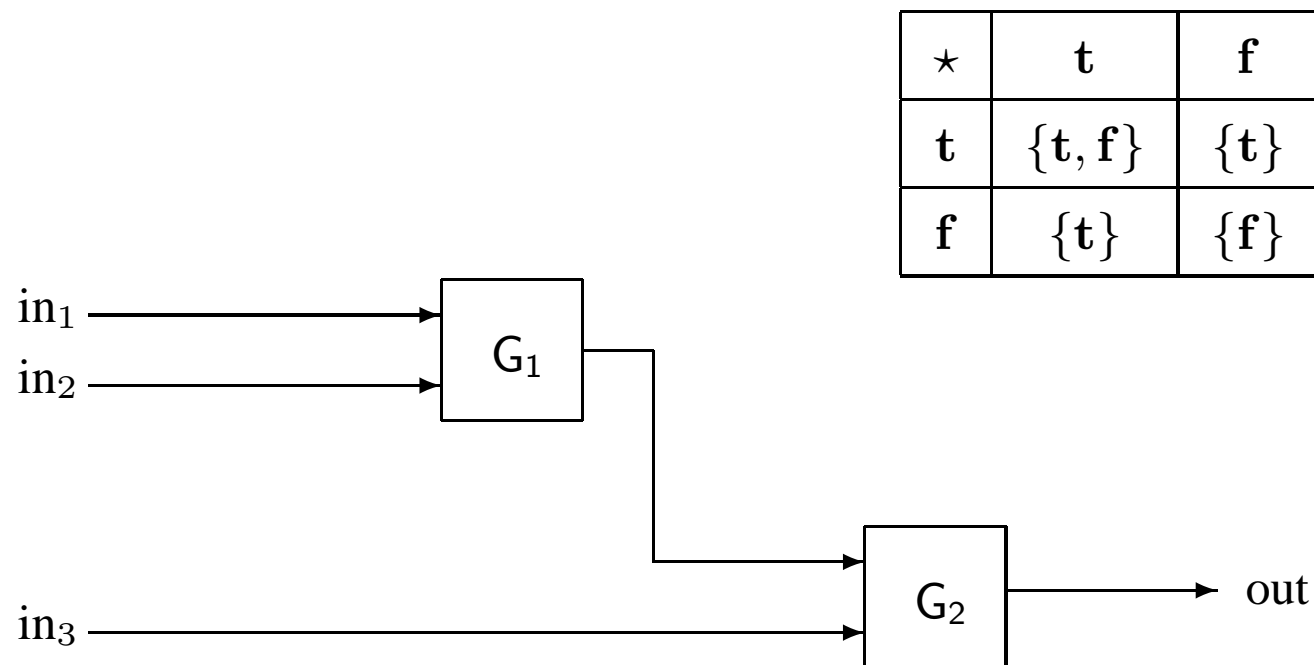
\star	t	f
t	{t, f}	{t}
f	{t}	{f}



Here the interpretation of the non-deterministic “truth-table” is **static**.

Deviation from expected behavior

- *Problems in the manufacturing process*
- *Disturbing noise sources, temperature, etc.*
- *Adversary operations*



Now the interpretation of the non-deterministic “truth-table” is **dynamic**.

Evaluation with Unknown Computation Models

- We are sending a formula $A \vee B$ for evaluation to a distant computer, and we do not know whether it performs parallel or sequential computation.
- As **parallel computation** can be described using 3-valued **Kleene connectives**, and **sequential computation** — using 3-valued **McCarthy connectives**, to reason about the result of evaluation we must use some **combination of both**.

\vee	f	I	t
f	{f}	{I}	{t}
I	{I}	{I}	{t, I}
t	{t}	{t}	{t}

Non-deterministic Matrices

A **non-deterministic matrix** (Nmatrix) for \mathcal{L} is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$:

- \mathcal{V} - the set of truth-values,
- \mathcal{D} - the set of designated truth-values,
- \mathcal{O} - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$ for every n -ary connective \diamond of \mathcal{L} .

Ordinary matrices correspond to the case when each $\tilde{\diamond}$ is a function taking singleton values only (then it can be treated as a function $\tilde{\diamond} : \mathcal{V}^n \rightarrow \mathcal{V}$).

Example 1: Two Truth-values

Let \mathcal{L} — a language over $\{\vee, \wedge, \supset, \neg\}$, $\mathcal{V} = \{\mathbf{f}, \mathbf{t}\}$, $\mathcal{D} = \{\mathbf{t}\}$.

Suppose that \vee, \wedge and \supset are interpreted classically, while \neg satisfies the law of excluded middle $\neg\varphi \vee \varphi$, but not necessarily the law of contradiction $\neg(\neg\varphi \wedge \varphi)$. This leads to the Nmatrix $\mathcal{M}^2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where \mathcal{O} is given by:

		$\tilde{\vee}$	$\tilde{\wedge}$	$\tilde{\supset}$		
\mathbf{t}	\mathbf{t}	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	\mathbf{t}	$\{\mathbf{t}, \mathbf{f}\}$
\mathbf{t}	\mathbf{f}	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	\mathbf{f}	$\{\mathbf{t}\}$
\mathbf{f}	\mathbf{t}	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$		
\mathbf{f}	\mathbf{f}	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{t}\}$		

Example 2: Three Truth-values

Two 3-valued Nmatrices with $\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}$, $\mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$

$$\mathcal{M}_L^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_L \rangle \quad \mathcal{M}_S^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_S \rangle$$

Standard interpretations of disjunction, conjunction and implication:

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \quad a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$

Negation is interpreted more liberally in \mathcal{M}_L^3 , and more strictly in \mathcal{M}_S^3 :

$\mathcal{M}_L^3 :$	<table style="border-collapse: collapse; margin: auto;"> <tr> <td style="border-right: 3px double black; padding: 5px;"></td> <td style="padding: 5px;">\approx</td> </tr> <tr> <td style="border-right: 3px double black; padding: 5px;">\mathbf{t}</td> <td style="padding: 5px;">$\{\mathbf{f}\}$</td> </tr> <tr> <td style="border-right: 3px double black; padding: 5px;">\mathbf{I}</td> <td style="padding: 5px;">\mathcal{V}</td> </tr> <tr> <td style="border-right: 3px double black; padding: 5px;">\mathbf{f}</td> <td style="padding: 5px;">$\{\mathbf{t}\}$</td> </tr> </table>		\approx	\mathbf{t}	$\{\mathbf{f}\}$	\mathbf{I}	\mathcal{V}	\mathbf{f}	$\{\mathbf{t}\}$
	\approx								
\mathbf{t}	$\{\mathbf{f}\}$								
\mathbf{I}	\mathcal{V}								
\mathbf{f}	$\{\mathbf{t}\}$								

$\mathcal{M}_S^3 :$	<table style="border-collapse: collapse; margin: auto;"> <tr> <td style="border-right: 3px double black; padding: 5px;"></td> <td style="padding: 5px;">\approx</td> </tr> <tr> <td style="border-right: 3px double black; padding: 5px;">\mathbf{t}</td> <td style="padding: 5px;">$\{\mathbf{f}\}$</td> </tr> <tr> <td style="border-right: 3px double black; padding: 5px;">\mathbf{I}</td> <td style="padding: 5px;">\mathcal{D}</td> </tr> <tr> <td style="border-right: 3px double black; padding: 5px;">\mathbf{f}</td> <td style="padding: 5px;">$\{\mathbf{t}\}$</td> </tr> </table>		\approx	\mathbf{t}	$\{\mathbf{f}\}$	\mathbf{I}	\mathcal{D}	\mathbf{f}	$\{\mathbf{t}\}$
	\approx								
\mathbf{t}	$\{\mathbf{f}\}$								
\mathbf{I}	\mathcal{D}								
\mathbf{f}	$\{\mathbf{t}\}$								

Dynamic and Static Valuations

- An \mathcal{M} -legal dynamic valuation v is any function from \mathcal{L} -formulas to \mathcal{V} which satisfies the condition:

$$v[\diamond(\psi_1, \dots, \psi_n)] \in \tilde{\diamond}[v[\psi_1], \dots, v[\psi_n]]$$

- An \mathcal{M} -legal static valuation v satisfies in addition the following *compositionality principle*:

$$v[\diamond(\psi_1, \dots, \psi_n)] = v[\diamond(\varphi_1, \dots, \varphi_n)] \text{ whenever } v[\psi_i] = v[\varphi_i] \\ \text{for } 1 \leq i \leq n$$

- Unlike the static semantics, the dynamic semantics is non-truth-functional.
- *In the deterministic case there is no difference between static and dynamic valuations.*

Dynamic and Static Valuations Explained

Two possible ways to choose the truth-value of $\diamond(\psi_1, \dots, \psi_n)$:

- **Dynamic (online) computation:** separately for every formula.
- **Static (offline) choice:** separately for every connective \diamond , but uniformly for all formulas $\diamond(\psi_1, \dots, \psi_n)$.

As opposed to static valuations, dynamic valuations need not be truth-functional:

$$v_1(p_1) = v_1(p_2) = \mathbf{t} ; v_1(q_1) = v_1(q_2) = \mathbf{t} ; v_1(p_1 \star q_1) = v_1(p_2 \star q_2) = \mathbf{t}$$

$$v_2(p_1) = v_2(p_2) = \mathbf{t} ; v_2(q_1) = v_2(q_2) = \mathbf{t} ; v_2(p_1 \star q_1) = \mathbf{t} ; v_2(p_2 \star q_2) = \mathbf{f}$$

\star	\mathbf{t}	\mathbf{f}
\mathbf{t}	$\{\mathbf{t}, \mathbf{f}\}$	$\{\mathbf{t}\}$
\mathbf{f}	$\{\mathbf{t}\}$	$\{\mathbf{f}\}$

Non-deterministic Semantics

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for \mathcal{L} .

- A valuation v in \mathcal{M} is a **model** of a formula ψ if $v[\psi] \in \mathcal{D}$.
A valuation v is a model of a set of formulas Γ if it is a model of every formula in Γ .
- $\Gamma \vdash_{\mathcal{M}}^d \psi$ if every dynamic model v in \mathcal{M} of Γ is a model of ψ .
- $\Gamma \vdash_{\mathcal{M}}^s \psi$ if every static model v in \mathcal{M} of Γ is a model of ψ .
- For every Nmatrix \mathcal{M} , **both $\vdash_{\mathcal{M}}^s$ and $\vdash_{\mathcal{M}}^d$ define logics**.
- A logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is **dynamically (statically) sound** for an Nmatrix \mathcal{M} for \mathcal{L} if $\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}^d$ ($\vdash_{\mathbf{L}} \subseteq \vdash_{\mathcal{M}}^s$). \mathbf{L} is **dynamically (statically) complete** for \mathcal{M} if $\vdash_{\mathcal{M}}^d \subseteq \vdash_{\mathbf{L}}$ ($\vdash_{\mathcal{M}}^s \subseteq \vdash_{\mathbf{L}}$). \mathcal{M} is **dynamically (statically) characteristic** for \mathbf{L} if $\vdash_{\mathcal{M}}^d = \vdash_{\mathbf{L}}$ ($\vdash_{\mathcal{M}}^s = \vdash_{\mathbf{L}}$).

The Nmatrix \mathcal{M}^2 Revisited

		$\tilde{\vee}$	$\tilde{\wedge}$	$\tilde{\supset}$		
t	t	{t}	{t}	{t}		
t	f	{t}	{f}	{f}	t	{t, f}
f	t	{t}	{f}	{t}	f	{t}
f	f	{f}	{f}	{t}		

- Any dynamic valuation satisfies $\neg\varphi \vee \varphi$ but not necessarily $\varphi \supset \neg\neg\varphi$.
- Any static valuation satisfies both $\neg\varphi \vee \varphi$ and $\varphi \supset \neg\neg\varphi$ (it admits only two interpretations of \neg : the classical one and $\lambda x.t$). However, it does not satisfy $\neg\varphi \wedge \varphi \supset \psi$ (take $\tilde{\neg} = \lambda x.t$, and let $v(\varphi) = \mathbf{t}$, $v(\psi) = \mathbf{f}$).
- The dynamic semantics and the static semantics generate two different **paraconsistent** logics, known as *CLuN* and *CAR*.

Reminder: The System GCPL

 $(\alpha) \psi \Rightarrow \psi$

$$(Weakening) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

$$(Cut) \quad \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta}$$

$$(\neg \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \neg) \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi}$$

$$(\supset \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow \supset) \quad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$$

$$(\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}$$

$$(\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

Reminder: The System HCL⁺

- **Axiom schemata:**

I1 $\varphi \supset (\psi \supset \varphi)$

I2 $(\varphi \supset \psi \supset \theta) \supset (\varphi \supset \psi) \supset (\varphi \supset \theta)$

I3 $((\psi \supset \varphi) \supset \psi) \supset \psi$

C1 $\varphi \wedge \psi \supset \varphi$

C2 $\varphi \wedge \psi \supset \psi$

C3 $\varphi \supset (\psi \supset \varphi \wedge \psi)$

D1 $\varphi \supset \varphi \vee \psi$

D2 $\psi \supset \varphi \vee \psi$

D3 $(\varphi \supset \theta) \supset (\psi \supset \theta) \supset (\varphi \vee \psi \supset \theta)$

- **Inference Rule:**

$$\frac{\psi \quad \psi \supset \varphi}{\varphi} MP$$

The Logic $CLuN$

- A Gentzen-style system sound and complete for the dynamic semantics of \mathcal{M}_2 is obtained by deleting from GCPL the rule $(\neg \Rightarrow)$, and leaving the single negation rule $(\Rightarrow \neg)$, i.e.:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi}$$

- As the above rule corresponds to **N3**: $\neg \varphi \vee \varphi$, a Hilbert-style system sound and complete for $\vdash_{\mathcal{M}_2}^d$ is obtained by adding **N3** to the system HCL^+ for positive classical logic.
- Since the above systems are respectively the standard Gentzen- and Hilbert-style systems used for the paraconsistent logic $CLuN$ of D. Batens, the logic corresponding to the dynamic semantics of \mathcal{M}_2 is indeed $CLuN$.

The Logic CAR

- A Hilbert-style system sound and complete for the static semantics of \mathcal{M}_2 is obtained by supplementing HCL^+ with **N3** and the axiom scheme $\varphi \supset (\neg\varphi \supset \neg\psi)$
- As the above system is equivalent to the original Hilbert-style system for the paraconsistent logic CAR , the logic corresponding to the static semantics of \mathcal{M}_2 is CAR .

\mathcal{M}_L^3 and \mathcal{M}_S^3 Revisited

$$\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \quad \mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$$

$$\mathcal{M}_L^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_L \rangle \quad \mathcal{M}_S^3 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_S \rangle$$

$$a \widetilde{\vee} b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \quad a \widetilde{\wedge} b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a \widetilde{\supset} b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$

$$\mathcal{M}_L^3 :$$

		\approx
\mathbf{t}		$\{\mathbf{f}\}$
\mathbf{I}		\mathcal{V}
\mathbf{f}		$\{\mathbf{t}\}$

$$\mathcal{M}_S^3 :$$

		\approx
\mathbf{t}		$\{\mathbf{f}\}$
\mathbf{I}		\mathcal{D}
\mathbf{f}		$\{\mathbf{t}\}$

Proof System for Dynamic Semantics of \mathcal{M}_L^3 and \mathcal{M}_S^3

- Let the Gentzen-style system \mathbf{GC}_{min} be obtained by replacing the rule $(\neg \Rightarrow)$ in GCPL with the rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$$

It can be shown that \mathbf{GC}_{min} is sound and complete for the dynamic semantics of both \mathcal{M}_L^3 and \mathcal{M}_S^3 .

- Thus the dynamic semantics of the Nmatrices \mathcal{M}_L^3 and \mathcal{M}_S^3 generate the same logic.

The Logic C_{\min}

- As the negation rules

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Rightarrow \Delta, \neg\varphi}$$

of GC_{\min} translate to **N2** ($\neg\neg\varphi \supset \varphi$) and **N1** ($\neg\varphi \vee \varphi$), respectively, GC_{\min} is equivalent to the system HC_{\min} obtained by adding the above axiom schemes to HCL^+ .

- The system HC_{\min} represents the **basic paraconsistent logic** C_{\min} . *That logic was originally defined in terms of the proof systems only, without any semantics.*

An Application: Decidability of C_{\min}

- By the foregoing, the system HC_{\min} is sound and complete for the dynamic semantics of both \mathcal{M}_L^3 and \mathcal{M}_S^3 . In consequence, \mathcal{M}_L^3 and \mathcal{M}_S^3 provide sound and complete semantics for the logic C_{\min} .
- This fact can be used for showing that some formulas are, or are not, theorems of C_{\min} – by showing that they are, or resp. are not, valid in the dynamic semantics of \mathcal{M}_L^3 (or \mathcal{M}_S^3). Thus $\varphi \supset \neg\neg\varphi$ is not derivable in C_{\min} , since $\not\vdash_{\mathcal{M}_L^3} \varphi \supset \neg\neg\varphi$.
To see this, take $v[\varphi] = \mathbf{I}$ and $v[\neg\varphi] = \mathbf{t}$. Then v is a legal dynamic valuation in \mathcal{M}_L^3 and $v[\neg\neg\varphi] \in \sim[\mathbf{t}] = \{\mathbf{f}\}$, whence $v[\varphi \supset \neg\neg\varphi] \in \mathbf{I} \tilde{\supset} \mathbf{f} = \{\mathbf{f}\}$ and so $\not\vdash_{\mathcal{M}_L^3} \varphi \supset \neg\neg\varphi$.
- It follows that C_{\min} is **decidable**.

Information Sources Logic: Generalization of Belnap's Logic

A processor P collects information from some sources, and assigns to a formula A the value:

- $\mathbf{t} = \{1\}$, if P has information that A is true, but no information that A is false.
- $\mathbf{f} = \{0\}$, if P has information that A is false, but no information that A is true.
- $\mathbf{I} = \{0, 1\}$, if P has both information that A is false and information that A is true.
- $\perp = \emptyset$, if P has no information on A .

Information Sources Logic

Assumption 1: The sources can provide information on any formulas (also complex ones), but not necessarily on all of them,

Assumption 2: P respects the classical truth-tables, i.e.:

1. P ascribes 1(0) to $\neg\varphi$ iff it ascribes 0 (1) to φ
2. P ascribes 1 to $\varphi \vee \psi$ if it ascribes 1 to either φ or ψ
3. P ascribes 0 to $\varphi \vee \psi$ iff it ascribes 0 to both φ and ψ
4. P ascribes 1 to $\varphi \wedge \psi$ iff it ascribes 1 to both φ and ψ
5. P ascribes 0 to $\varphi \wedge \psi$ if it ascribes 0 to either φ or ψ

This framework generalizes Belnap's one by allowing the sources to give information on complex formulas too, and, in opposition to the former, **cannot be captured using deterministic matrices**.

Assignment of Values to Formulas by the Processor

The processor P assigns to a formula ψ a value $v(\psi)$ in the set

$$\perp = \emptyset \quad \mathbf{f} = \{0\} \quad \mathbf{I} = \{0, 1\} \quad \mathbf{t} = \{1\}$$

whereby $1 \in v[\psi]$ (resp. $0 \in v[\psi]$) iff either one of the sources tells P that ψ is true (resp. false), or P concludes the latter from the information it already has and from Rules 1-5 it obeys.

Thus if $v[\varphi] = v[\psi] = \mathbf{f} = \{0\}$, then surely $0 \in v[\varphi \vee \psi]$ by Rule 3. However, if some source says $\varphi \vee \psi$ is true, then P will ascribe 1 to $\varphi \vee \psi$, making $v[\varphi \vee \psi] = \{0, 1\} = \mathbf{I}$. Otherwise, we have $v[\varphi \vee \psi] = \{0\} = \mathbf{f}$. Summing up, $v[\varphi \vee \psi] \in \{\mathbf{f}, \mathbf{I}\}$ — which shows that the interpretation of \vee must be non-deterministic.

Nmatrix \mathcal{M}_I^4 for the Information Sources Logic

$$\mathcal{V} = \{\mathbf{f}, \perp, \mathbf{I}, \mathbf{t}\} \quad \mathcal{D} = \{\mathbf{t}, \mathbf{I}\}$$

\approx	\mathbf{f}	\perp	\mathbf{I}	\mathbf{t}
	\mathbf{t}	\perp	\mathbf{I}	\mathbf{f}

$\tilde{\vee}$	\mathbf{f}	\perp	\mathbf{I}	\mathbf{t}
\mathbf{f}	$\{\mathbf{f}, \mathbf{I}\}$	$\{\mathbf{t}, \perp\}$	$\{\mathbf{I}\}$	$\{\mathbf{t}\}$
\perp	$\{\mathbf{t}, \perp\}$	$\{\mathbf{t}, \perp\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$
\mathbf{I}	$\{\mathbf{I}\}$	$\{\mathbf{t}\}$	$\{\mathbf{I}\}$	$\{\mathbf{t}\}$
\mathbf{t}	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$	$\{\mathbf{t}\}$

$\tilde{\wedge}$	\mathbf{f}	\perp	\mathbf{I}	\mathbf{t}
\mathbf{f}	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$
\perp	$\{\mathbf{f}\}$	$\{\mathbf{f}, \perp\}$	$\{\mathbf{f}\}$	$\{\mathbf{f}, \perp\}$
\mathbf{I}	$\{\mathbf{f}\}$	$\{\mathbf{f}\}$	$\{\mathbf{I}\}$	$\{\mathbf{I}\}$
\mathbf{t}	$\{\mathbf{f}\}$	$\{\mathbf{f}, \perp\}$	$\{\mathbf{I}\}$	$\{\mathbf{t}, \mathbf{I}\}$

Gentzen System for the Information Sources Logic

Standard axiom, cut, weakening and the rules:

$$(\neg\neg \Rightarrow) \quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$$

$$(\Rightarrow \neg\neg) \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$(\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\neg\vee \Rightarrow) \quad \frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$$

$$(\Rightarrow \neg\vee) \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$$

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta,}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$$

$$(\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$(\Rightarrow \neg\wedge) \quad \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$$

Power of Dynamic Semantics

- Let \mathcal{M} be a 2-valued Nmatrix which has at least one proper non-deterministic operation. **There is no finite family of finite ordinary matrices C , such that $\vdash_{\mathcal{M}}^d = \vdash_C$.**
- If in addition \mathcal{M} includes the classical implication, then there is even no finite family of finite ordinary matrices C such that $\vdash_{\mathcal{M}}^d \psi$ iff $\vdash_C \psi$.

Expressive Power of Static Semantics

- For every (finite) Nmatrix \mathcal{M} there is a (finite) family C of matrices such that $\vdash_{\mathcal{M}}^s = \vdash_C$.
- Thus only the expressive power of the dynamic semantics based on Nmatrices is stronger than that of ordinary matrices.

Accordingly, in the rest of this course our main focus will be on the dynamic semantics, and we shall write simply $\vdash_{\mathcal{M}}$ instead of $\vdash_{\mathcal{M}}^d$.

Analycity, Decidability and Compactness

An obvious, yet crucial fact: any partial \mathcal{M} -legal valuation defined on a set closed under subformulas can be extended to a full \mathcal{M} -legal valuation.

If \mathcal{M} is **finite** then this entails that $\vdash_{\mathcal{M}}$ is:

- **Decidable**
- **Finitary** (the compactness theorem obtains)

Refinements

Let $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ be Nmatrices for \mathcal{L} .

- \mathcal{M}_1 is a **refinement** of \mathcal{M}_2 if:
 1. $\mathcal{V}_1 \subseteq \mathcal{V}_2$
 2. $\mathcal{D}_1 = \mathcal{D}_2 \cap \mathcal{V}_1$
 3. $\tilde{\diamond}_{\mathcal{M}_1}(x_1, \dots, x_n) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(x_1, \dots, x_n)$ for every n -ary connective \diamond of \mathcal{L} and every $x_1, \dots, x_n, y \in \mathcal{V}_1$.
- If \mathcal{M}_1 is a refinement of \mathcal{M}_2 then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.

Examples of Refinements

- \mathcal{M}_S^3 is a refinement of \mathcal{M}_L^3 .
- The classical two-valued matrix is a refinement of \mathcal{M}_S^3 .
- The classical two-valued matrix is also a refinement of \mathcal{M}^2 .

$\mathcal{M}_L^3 :$

		\approx
t		{f}
I		\mathcal{V}
f		{t}

$\mathcal{M}_S^3 :$

		\approx
t		{f}
I		\mathcal{D}
f		{t}

$\mathcal{M}^2 :$

		\approx
t		{t, f}
f		{t}

Expansions

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for \mathcal{L} . Assume that F is a function that assigns to each $x \in \mathcal{V}$ a nonempty set $F[x]$ so that $F[x_1] \cap F[x_2] = \emptyset$ if $x_1 \neq x_2$. The **F -expansion** of \mathcal{M} is the following Nmatrix $\mathcal{M}_F = \langle \mathcal{V}_F, \mathcal{D}_F, \mathcal{O}_F \rangle$:

- $\mathcal{V}_F = \bigcup_{x \in \mathcal{V}} F[x]$
- $\mathcal{D}_F = \bigcup_{x \in \mathcal{D}} F[x]$
- $\tilde{\diamond}_{\mathcal{M}_F}[y_1, \dots, y_n] = F[\tilde{\diamond}_{\mathcal{M}}[x_1, \dots, x_n]]$ whenever \diamond is an n -ary connectives of \mathcal{L} , and $x_i \in \mathcal{V}$, $y_i \in F[x_i]$ for every $1 \leq i \leq n$.

If \mathcal{M}_1 is an expansion of \mathcal{M}_2 then $\vdash_{\mathcal{M}_2} = \vdash_{\mathcal{M}_1}$.

Example

Let $F[\mathbf{t}] = \{\mathbf{t}, \mathbf{I}\}$ and $F[\mathbf{f}] = \{\mathbf{f}\}$. **The $\{\vee, \wedge, \supset\}$ -part of \mathcal{M}_L^3 (or \mathcal{M}_S^3) is obtained as the F -expansion of the positive part of the classical two-valued matrix.**

$$\mathcal{V} = \{\mathbf{f}, \mathbf{I}, \mathbf{t}\}, \quad \mathcal{D} = \{\mathbf{I}, \mathbf{t}\}$$

$$a\widetilde{\vee}b = \begin{cases} \mathcal{D} & \text{if either } a \in \mathcal{D} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a = b = \mathbf{f} \end{cases} \quad a\widetilde{\wedge}b = \begin{cases} \mathcal{D} & \text{if } a, b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if either } a = \mathbf{f} \text{ or } b = \mathbf{f} \end{cases}$$

$$a\widetilde{\supset}b = \begin{cases} \mathcal{D} & \text{if either } a = \mathbf{f} \text{ or } b \in \mathcal{D} \\ \{\mathbf{f}\} & \text{if } a \in \mathcal{D} \text{ and } b = \mathbf{f} \end{cases}$$