

INTRODUCTION TO
ABSTRACT CATEGORIAL
GRAMMARS

THIRD COURSE

ABSTRACT CATEGORIAL GRAMMAR

$s \in A_1$

(THE STARTING SYMBOL)

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$$\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$$

WHERE:

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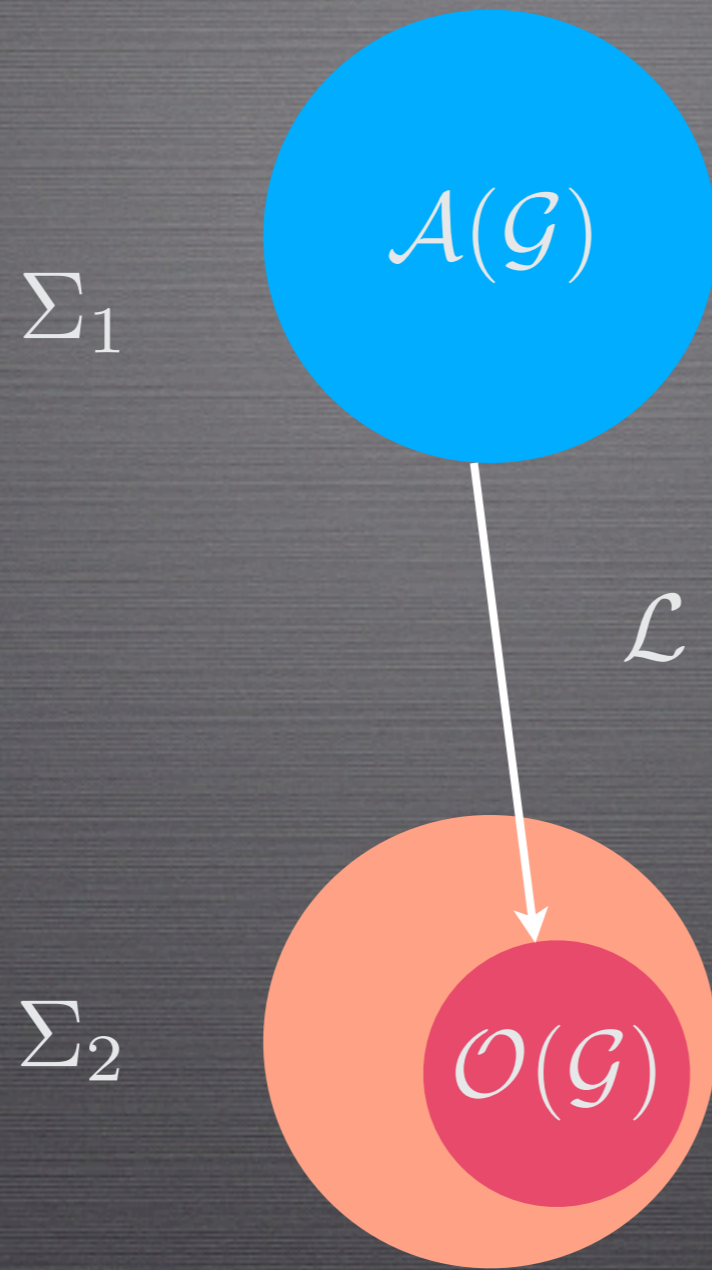
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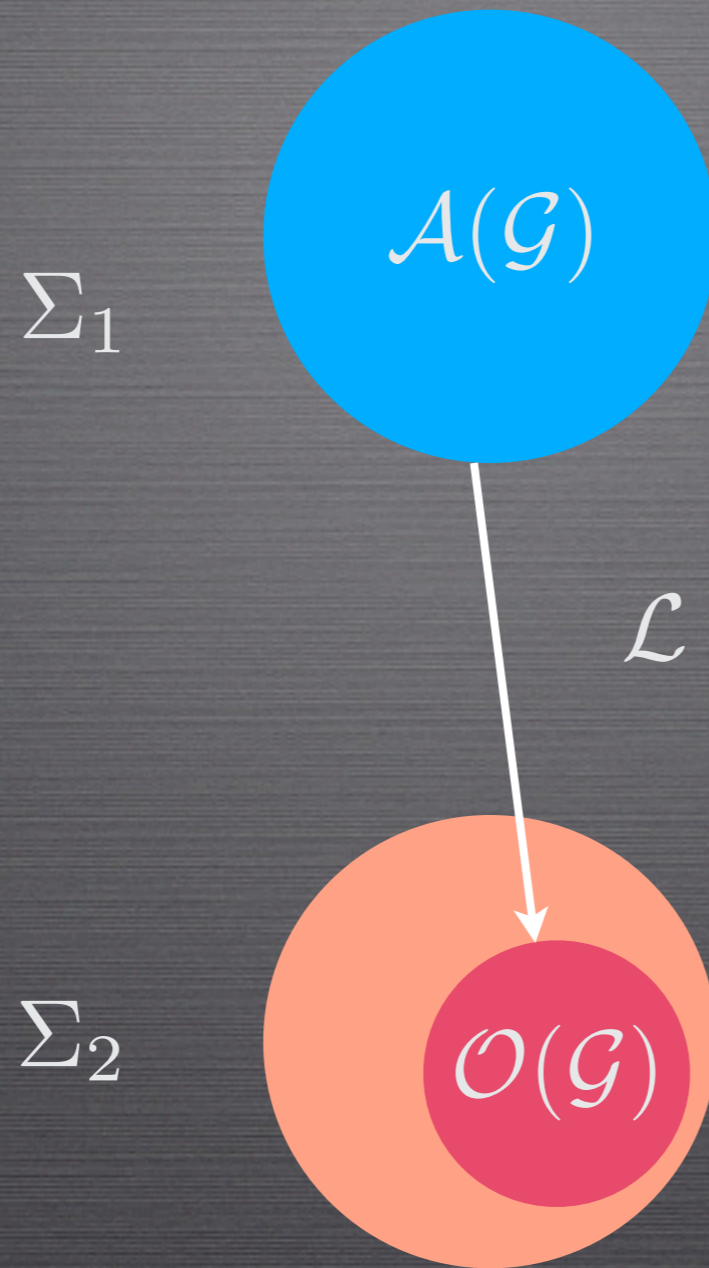
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A CLASSIFICATION OF ACGS

TWO PARAMETERS

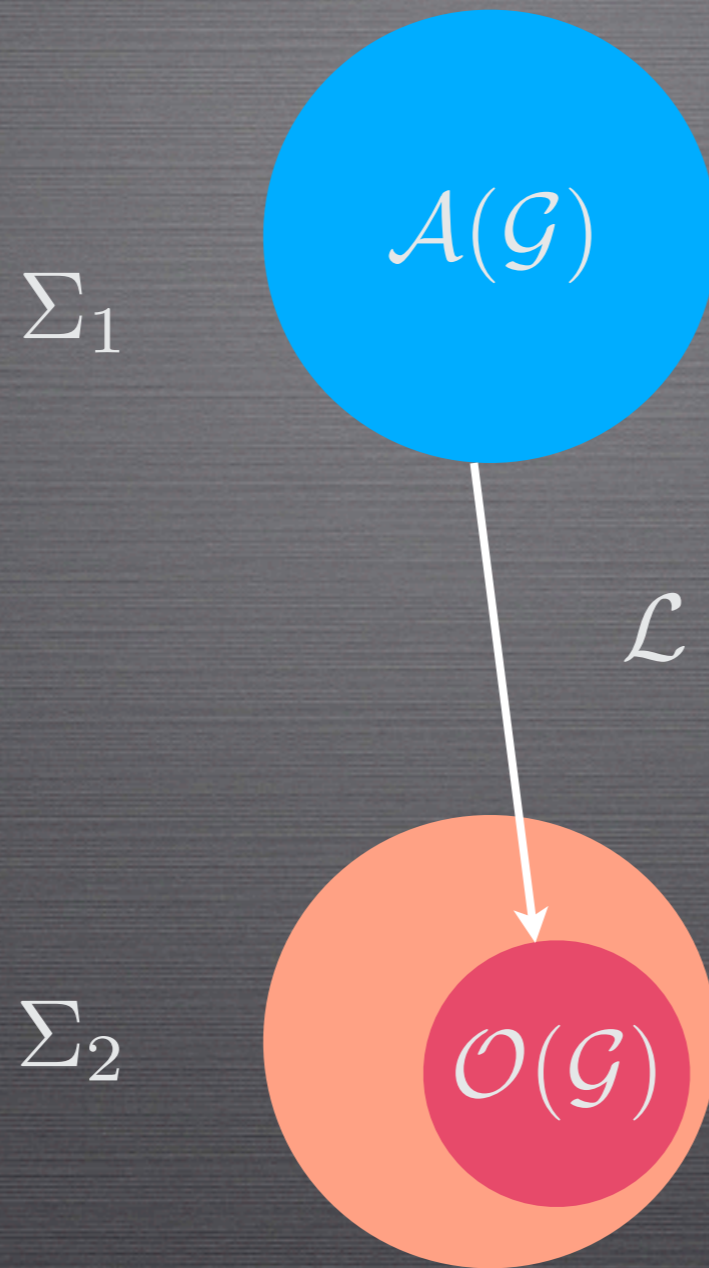


TWO PARAMETERS



THE COMPLEXITY
OF THE ABSTRACT
STRUCTURES

TWO PARAMETERS



THE COMPLEXITY
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THE COMPLEXITY
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ORDER OF TYPES

$\text{ord}(\alpha) = 1$ when α is atomic

$\text{ord}(\alpha \rightarrow \beta) = \max(\text{ord}(\alpha) + 1, \text{ord}(\beta))$

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$(o \rightarrow o) \rightarrow (((o \rightarrow o) \rightarrow o) \rightarrow o) \rightarrow o$



THE ORDER OF A SIGNATURE

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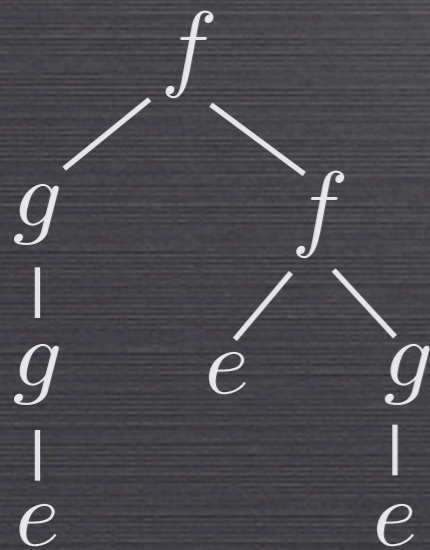
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SECOND ORDER CASE



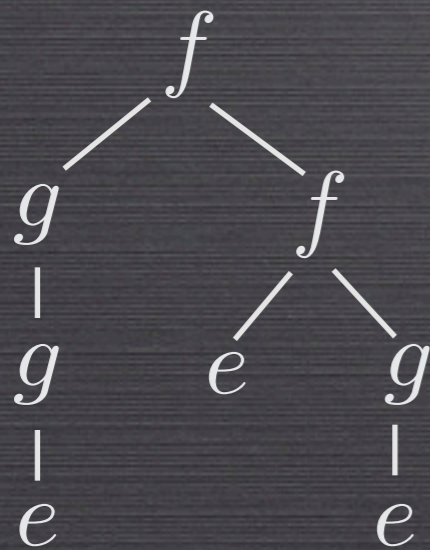
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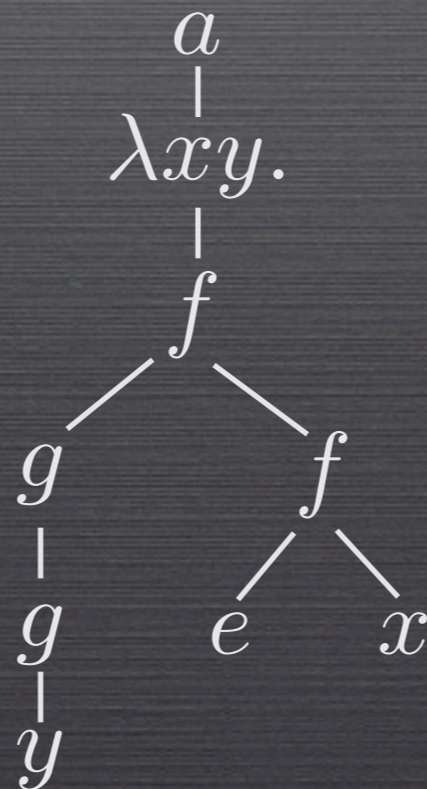
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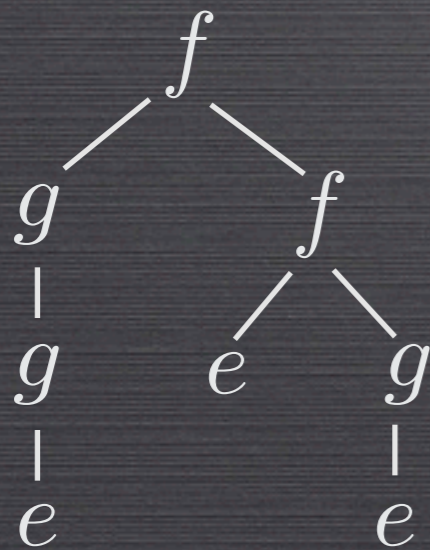
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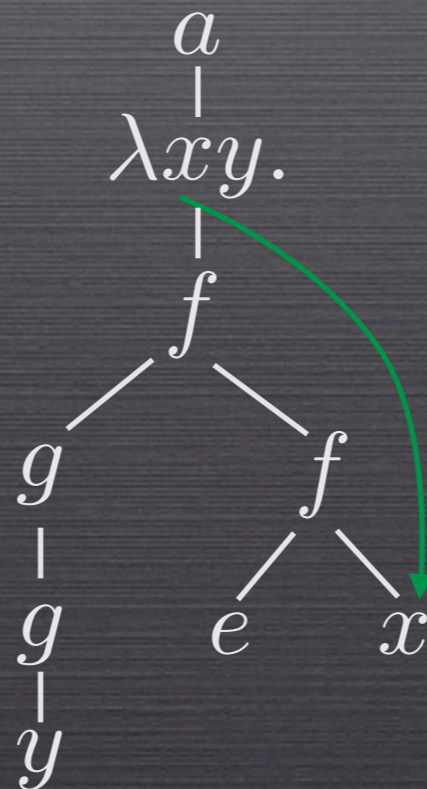
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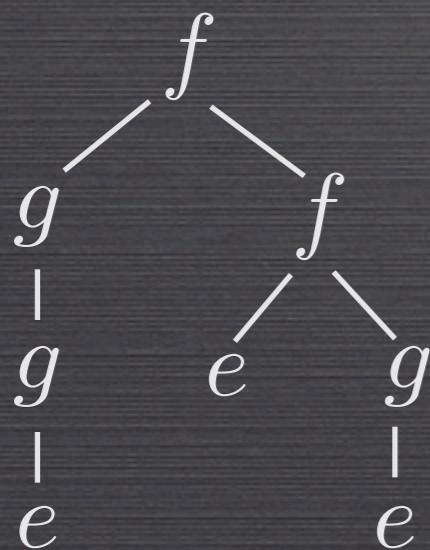
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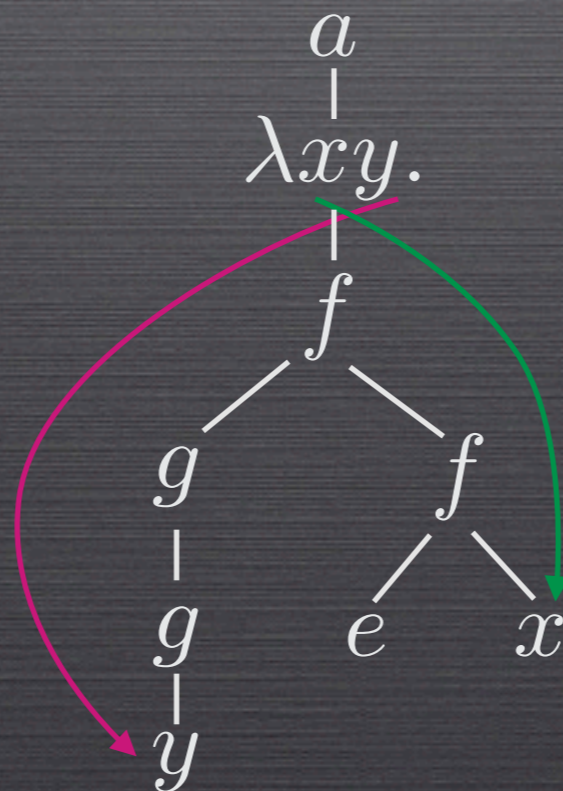
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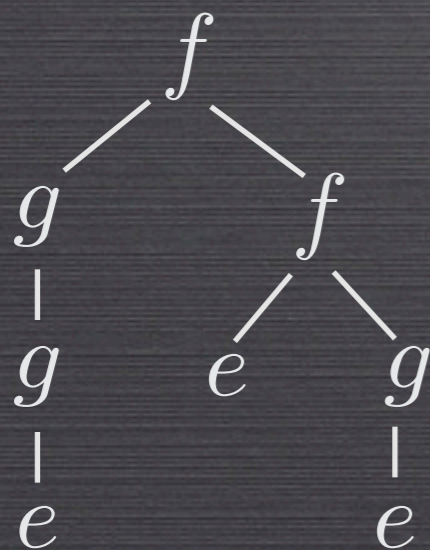
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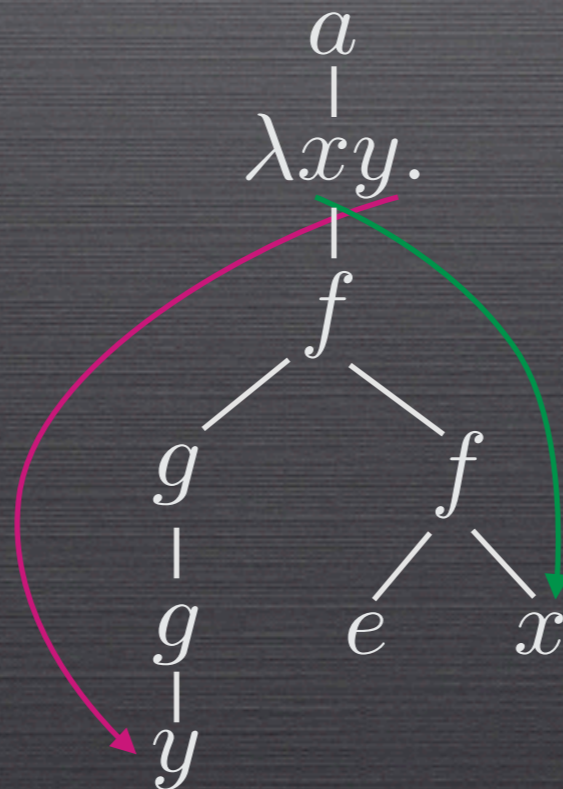
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FIFTH ORDER
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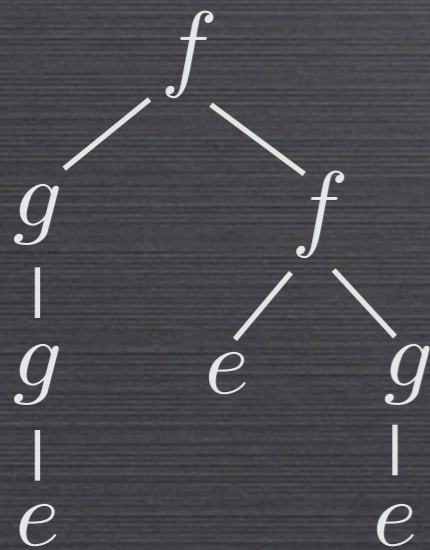
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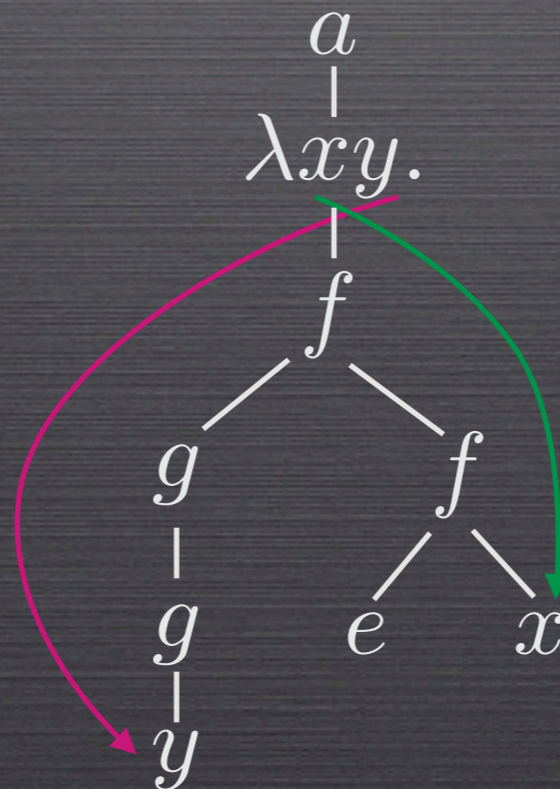
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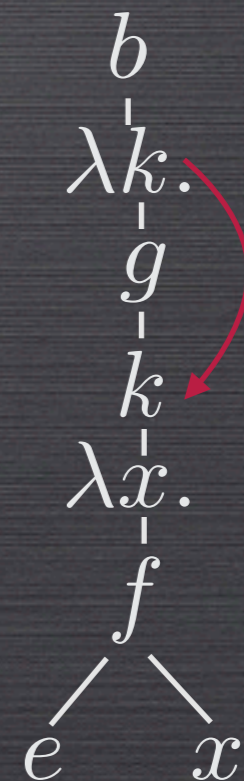
THIRD ORDER

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FIFTH ORDER

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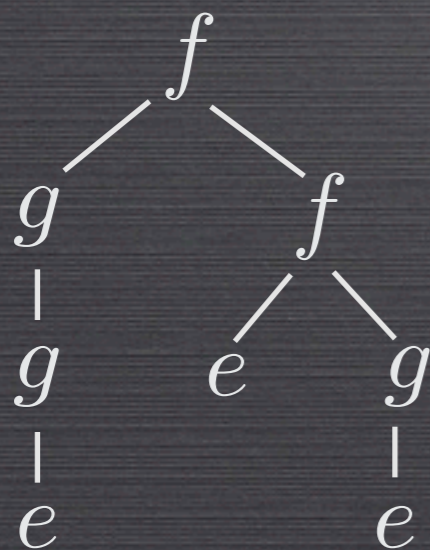
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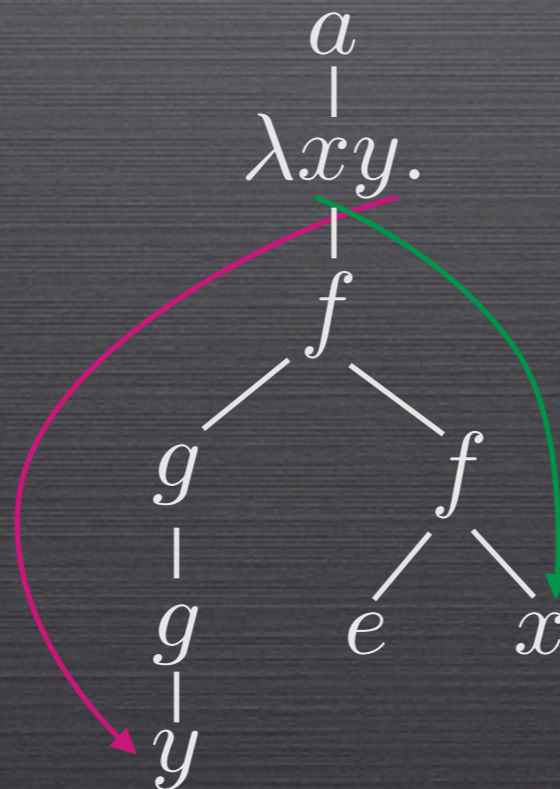
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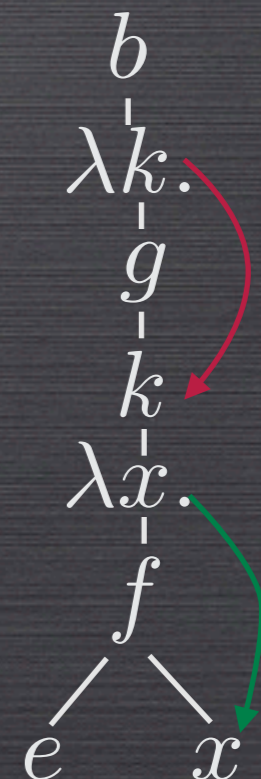
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THE COMPLEXITY OF A LEXICON

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$$\text{comp}(\mathcal{L}) = \max_{\alpha \in A_1} (\text{ord}(\mathcal{L}(\alpha)))$$

ORDER AND COMPLEXITY OF A GRAMMAR

LET $\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$

$$\text{order}(\mathcal{G}) = \text{ord}(\Sigma_1)$$

$$\text{complexity}(\mathcal{G}) = \text{comp}(\mathcal{L})$$

ABSTRACT CATEGORIAL HIERARCHY

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$$\mathbf{G}(m, n) = \{\mathcal{G} \mid \text{order}(\mathcal{G}) \leq m \text{ and } \text{complexity}(\mathcal{G}) \leq n\}$$

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$$\mathbf{G}(m, n) = \{\mathcal{G} \mid \text{order}(\mathcal{G}) \leq m \text{ and } \text{complexity}(\mathcal{G}) \leq n\}$$

$$\mathbf{L}(m, n) = \{\mathcal{O}(\mathcal{G}) \mid \mathcal{G} \in \mathbf{G}(m, n)\}$$

PROPOSITION:

FOR ALL $m, n \geq 1$, $\mathbf{L}(m, n + 1) \subset \mathbf{L}(m + 1, n)$

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SUPPOSE $\mathcal{L}(\gamma) = \alpha \rightarrow \beta$

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$$\mathcal{G}' = \langle \Sigma'_1, \Sigma_2, \mathcal{L}', s \rangle$$

$$\mathcal{L}'(\delta_1) = \alpha$$

$$\mathcal{L}'(c) = \mathcal{L}(c)$$

$$\mathcal{L}'(\delta_2) = \beta$$

$$\mathcal{L}'(\beta) = \mathcal{L}(\beta) \text{ when } \beta \in A_1$$

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$$\tau'_1(c) = \tau_1(c)[\alpha \rightarrow \beta := \gamma] \text{ when } c \in C_1$$

$$\tau'_1(\text{enc}) = (\alpha \rightarrow \beta) \rightarrow \gamma \quad \tau'_1(\text{tr}) = \gamma \rightarrow \alpha \rightarrow \beta$$

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$$\mathcal{G}' = \langle \Sigma'_1, \Sigma_2, \mathcal{L}', s \rangle \quad \mathcal{L}'(\delta) := \mathcal{L}(\delta) \text{ when } \delta \in A_1$$

$$\mathcal{L}'(\gamma) := \mathcal{L}'(\alpha) \rightarrow \mathcal{L}'(\beta)$$

$$\mathcal{L}'(c) := \mathcal{L}(c) \text{ when } c \in C_1$$

$$\mathcal{L}'(\text{tr}) := \lambda x. \lambda y. (x y)$$

$$\mathcal{L}'(\text{enc}) := \lambda x. x$$

HIERARCHIES OF INTEREST:

$L(1, n)$

$L(2, n)$

$L(3, n)$

L(1, *n*)

$\mathbf{L}(1, n)$

LET $\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$ **BE IN** $\mathbf{G}(1, n)$

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THE ONLY CLOSED TERMS OF TYPE s ARE CONSTANTS

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THE ONLY CLOSED TERMS OF TYPE s ARE CONSTANTS

$\mathbf{L}(1, n)$ **IS THE CLASS OF FINITE LANGUAGES**

$L(2, n)$

**EXPRESSIVITY
ON STRINGS**

THE STRING LANGUAGES OF $\mathbf{L}(2, n)$

$\mathbf{L}_{string}(2, 1)$	REGULAR
$\mathbf{L}_{string}(2, 2)$	CONTEXT-FREE
$\mathbf{L}_{string}(2, 3)$	WELL-NESTED MILDLY CONTEXT SENSITIVE
$\mathbf{L}_{string}(2, 4)$	MILDLY CONTEXT SENSITIVE
$\mathbf{L}_{string}(2, 4 + n)$	$= \mathbf{L}_{string}(2, 4)$

CFLs $\subseteq \mathbf{L}_{string}(2, 2)$

LET $G = (N, T, R, s)$ BE A CFG

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$$X \rightarrow w_1 X_1 \dots w_n X_n w_{n+1}$$

$$r : X_1 \rightarrow \dots \rightarrow X_n \rightarrow X$$

CFLs $\subseteq \mathbf{L}_{string}(2, 2)$

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$$r : X_1 \rightarrow \dots \rightarrow X_n \rightarrow X$$

$$r := \lambda X_1 \dots X_n. w_1 + X_1 + \dots w_n + X_n + w_{n+1}$$

$\mathbf{L}_{string}(2, 2) \subseteq CFLs$

LET $\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$ **BE IN** $\mathbf{G}_{string}(2, 2)$

→ EVERY USEFULL TYPE VERIFIES

$$\mathcal{L}(\alpha) = \sigma \rightarrow \sigma$$

$L_{string}(2, 2) \subseteq CFLs$

LET $\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$ BE IN $G_{string}(2, 2)$

PROPOSITION

IF THERE IS AN ABSTRACT TYPE α IN Σ_1 SUCH THAT

$$\mathcal{L}(\alpha) = \sigma \rightarrow \dots \rightarrow \sigma \rightarrow \sigma \rightarrow \sigma$$

THEN IN A TERM OF TYPE s

THERE IS NO SUBTERM OF TYPE α

→ EVERY USEFULL TYPE VERIFIES

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LET $\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$ BE IN $\mathbf{G}_{string}(2, 2)$

GIVEN c AN ABSTRACT CONSTANT OF TYPE

$$\alpha_1 \rightarrow \cdots \rightarrow \alpha_n \rightarrow \alpha$$

WE HAVE

$$\mathcal{L}(c) = \lambda x_1 \dots x_n. w_1 + x_{\sigma(1)} + \cdots + w_n + x_{\sigma(n)} + w_{n+1}$$

$L_{string}(2, 2) \subseteq CFLs$

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$$\alpha \rightarrow w_1 \alpha_{\sigma(1)} \cdots w_n \alpha_{\sigma(n)} w_{n+1}$$

$$\mathbf{L}_{string}(2, 3) = \mathbf{WNMCSL}$$

VERY SIMILAR TO $\mathbf{L}_{string}(2, 2) = \mathbf{CFL}$

- WE FIRST CODE LINEAR MACRO GRAMMARS
- WE PROVE THAT THE USEFULL ABSTRACT TYPES ARE ONLY BUILT WITH THE *string* TYPE
- WE CONCLUDE

$$\mathbf{L}_{string}(2, n) = \mathbf{MCSL}$$

- $\mathbf{MCSL} = \mathbf{LCFRL}, \mathbf{MCFL}, \mathbf{SHR}, \text{Output}(\mathbf{DTWT}) \dots$
- **THERE ARE SEVERAL PROOFS OF $\mathbf{L}_{string}(2, n) = \mathbf{MCSL}$**
 - ▶ **IN SECOND ORDER ACG, THE HOMOMORPHISM ACTS LIKE A DTWT (SALVATI 06)**
 - ▶ **AN ENCODING OF SECOND ORDER ACGS AS HR GRAMMARS (KANAZAWA 07)**
 - ▶ **A PROOF USING INTERPOLATION (KANAZAWA, SALVATI 08).**

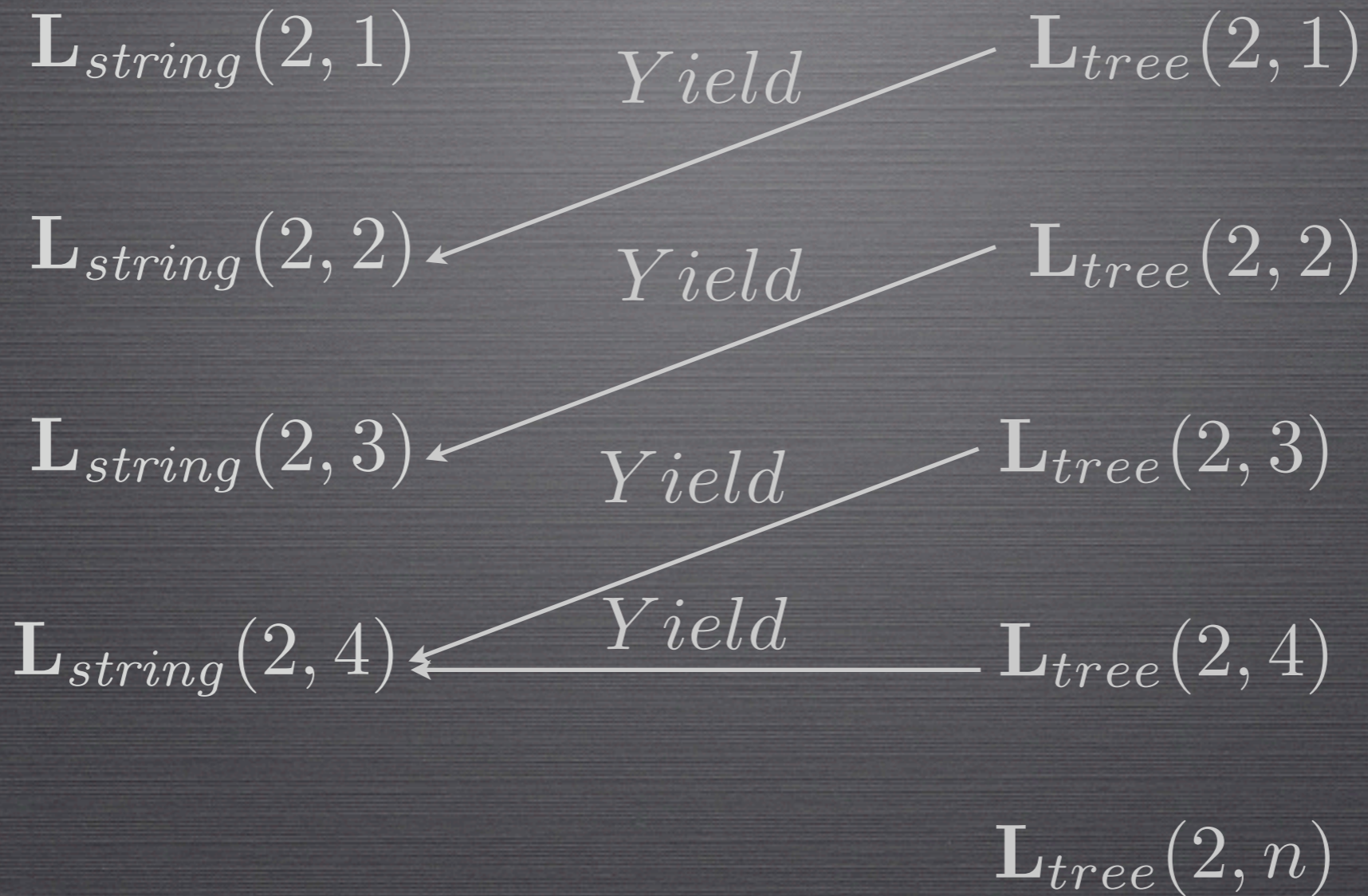
$L(2, n)$

**EXPRESSIVITY ON
TREES**

THE TREE LANGUAGES OF $L(2, n)$

$L_{tree}(2, 1)$	REGULAR
$L_{tree}(2, 2)$	LINEAR CONTEXT-FREE
$L_{tree}(2, 3)$?
$L_{tree}(2, 4)$	THR
$L_{tree}(2, n)$	$= L_{tree}(2, 4)$

TREES AND STRINGS



$L(2, n)$

**EXPRESSIVITY ON
TERMS**

LANGUAGES OF TERMS

- IT IS NOT KNOWN WHETHER THE HIERARCHY COLLAPSES IN THAT CASE
- THOSE LANGUAGES CAN BE FAITHFULLY REPRESENTED AS HR LANGUAGES (KANAZAWA 08).

L(3, *n*)

SEMILINEARITY

ALL THE LANGUAGES OF $L(2, n)$ ARE SEMILINEAR

THIS PROPERTY DOES NOT HOLD ANYMORE FOR
 $L(3, n)$