

INTRODUCTION TO
ABSTRACT CATEGORIAL
GRAMMARS

FOURTH COURSE

- PRELIMINARY NOTIONS:
 - ▶ COHERENCE THEOREM
 - ▶ CLOSURE PROPERTIES
 - ▶ DEFINITION OF MEMBERSHIP PROBLEMS
- COMPLEXITY OF MEMBERSHIP PROBLEMS:
 - ▶ $G(3, n)$
 - ▶ $G(2, n)$
- NON-LINEAR ACGS

THE COHERENCE THEOREM

PROPERTIES OF THE LINEAR LAMBDA-CALCULUS:

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- THE PRINCIPAL TYPE OF A LINEAR LAMBDA-TERM SPECIFIES UNIQUELY THIS TERM (COHERENCE)

LINEARITY AND COHERENCE

LET'S TAKE $\Sigma = (\{o\}, \{a\}, \{a : o \rightarrow o\})$

$\lambda x.a(a(x))$ HAS $o \rightarrow o$ AS PRINCIPAL TYPE

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BUT SO DOES $\lambda x.a(a(a(x)))$

RECOVERING LINEARITY

RELABELINGS: LEXICONS OF COMPLEXITY 1

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EXAMPLE:

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$$\begin{array}{ll} \mathcal{R}(o_i) & = o \\ \mathcal{R}(a_i) & = a \end{array} \quad \lambda x.a_1(a_2 x) \text{ IS THE ONLY TERM OF TYPE } o_3 \rightarrow o_1 \text{ IN } \Lambda_{\Sigma'}$$

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$$R(\lambda x.a_1(a_2 x)) = \lambda x.a(a x)$$

RECOVERING LINEARITY

PROPOSITION (COHERENCE):

FOR EVERY TERM t IN Λ_{Σ} THERE IS:

A SIGNATURE	Σ'
A RELABELING	$\mathcal{R} : \Sigma' \rightarrow \Sigma$
A TERM	t' IN $\Lambda_{\Sigma'}^{\alpha}$
VERIFYING	

RECOVERING LINEARITY

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$$\mathcal{R}(t') = t$$

INTERPRETING COHERENCE

$$a : o \rightarrow o$$

$$a_i : o_{i+1} \rightarrow o_i$$

$$\lambda x.a(a(a(a(a(x)))))$$

$$\lambda x.a_1(a_2(a_3(a_4(a_5(x)))) : o_6 \rightarrow o_1$$

$$o_1 \quad o_2 \quad o_3 \quad o_4 \quad o_5 \quad o_6$$

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STRING CONTEXTS:

$$\lambda s.a_i + \cdots + a_{i+k} + s + a_{i+k+l} + \cdots + a_{i+k+l+m} : \\ (o_{i+k+1} \rightarrow o_{i+k+l}) \rightarrow (o_{i+k+l+m} \rightarrow o_i)$$

CLOSURE PROPERTIES

CLOSURE UNDER INVERSE RELABELING

PROPOSITION:

LET $\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$ OF $\mathbf{G}(m, n)$

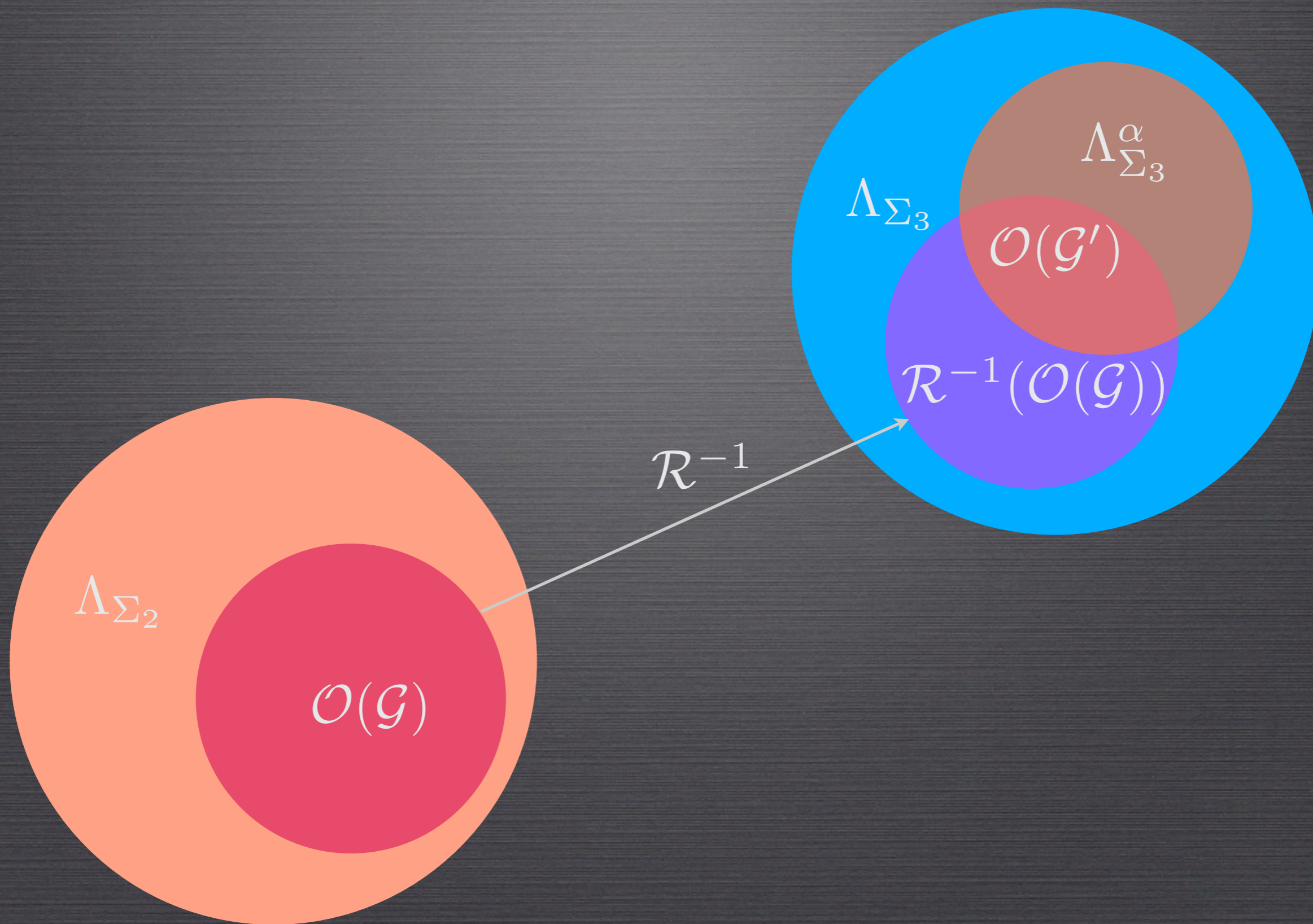
$\mathcal{R} : \Sigma_3 \rightarrow \Sigma_1$

α A TYPE IN Σ_3 SUCH THAT $\mathcal{R}(\alpha) = \mathcal{L}(s)$

THERE IS $\mathcal{G}' = \langle \Sigma'_1, \Sigma_3, \mathcal{L}', s' \rangle$ OF $\mathbf{G}(m, n)$ SUCH THAT

$$\mathcal{O}(\mathcal{G}') = \mathcal{R}^{-1}(\mathcal{O}(\mathcal{G})) \cap \Lambda_{\Sigma_3}^\alpha$$

A PICTORIAL VIEW



OTHER CLOSURE PROPERTIES

$L(m, n)$ IS CLOSED UNDER UNION

$L_{string}(m, n)$ IS CLOSED UNDER INTERSECTION
WITH REGULAR SETS OF STRINGS

$L_{tree}(m, n)$ IS CLOSED UNDER INTERSECTION
WITH REGULAR SETS OF TREES

MEMBERSHIP PROBLEMS

TWO KINDS OF PROBLEMS

UNIVERSAL MEMBERSHIP:

FIXED DATA: A CLASS OF GRAMMAR

INPUT: A GRAMMAR \mathcal{G} AND AN OBJECT t

OUTPUT: YES/NO WHEN $t \in L(\mathcal{G})/t \notin L(\mathcal{G})$

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	Membership	Universal Membership
CFGs/TAGs	P	P
Lambek Grammars	P	NP-complete
MCFGs	P	PSPACE-complete / EXPTIME-complete

COMPLEXITY OF

$G(3, n)$

MELL

THE EMPTINESS PROBLEM IN $G(3, n)$
IS TURING-EQUIVALENT TO PROOF SEARCH
IN MELL

SO IS THE MEMBERSHIP PROBLEM

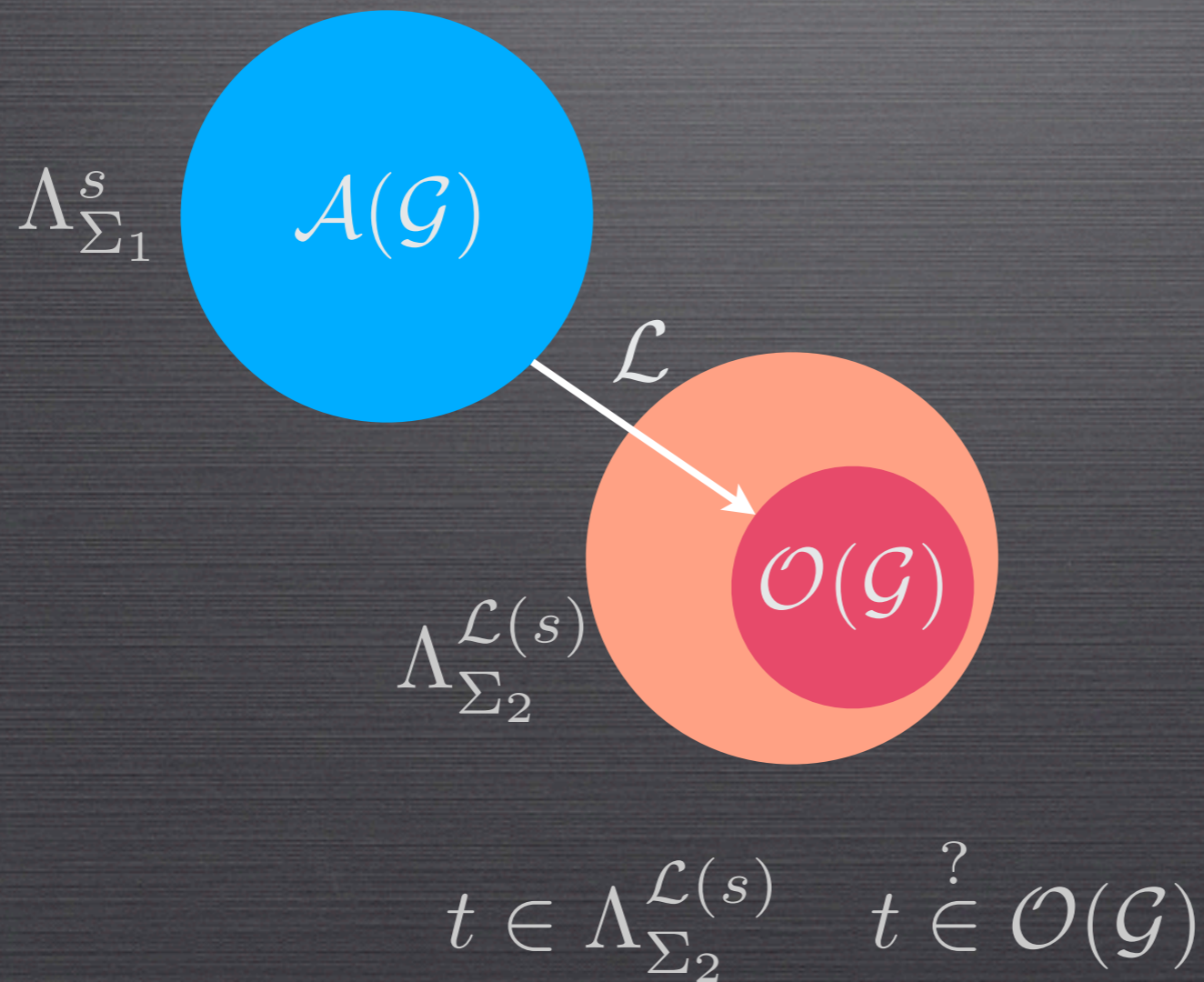
MELL

PROOF SEARCH IN MELL IS TURING-
EQUIVALENT TO THE EMPTINESS
PROBLEM OF $\Lambda_{\Sigma}^{\alpha}$

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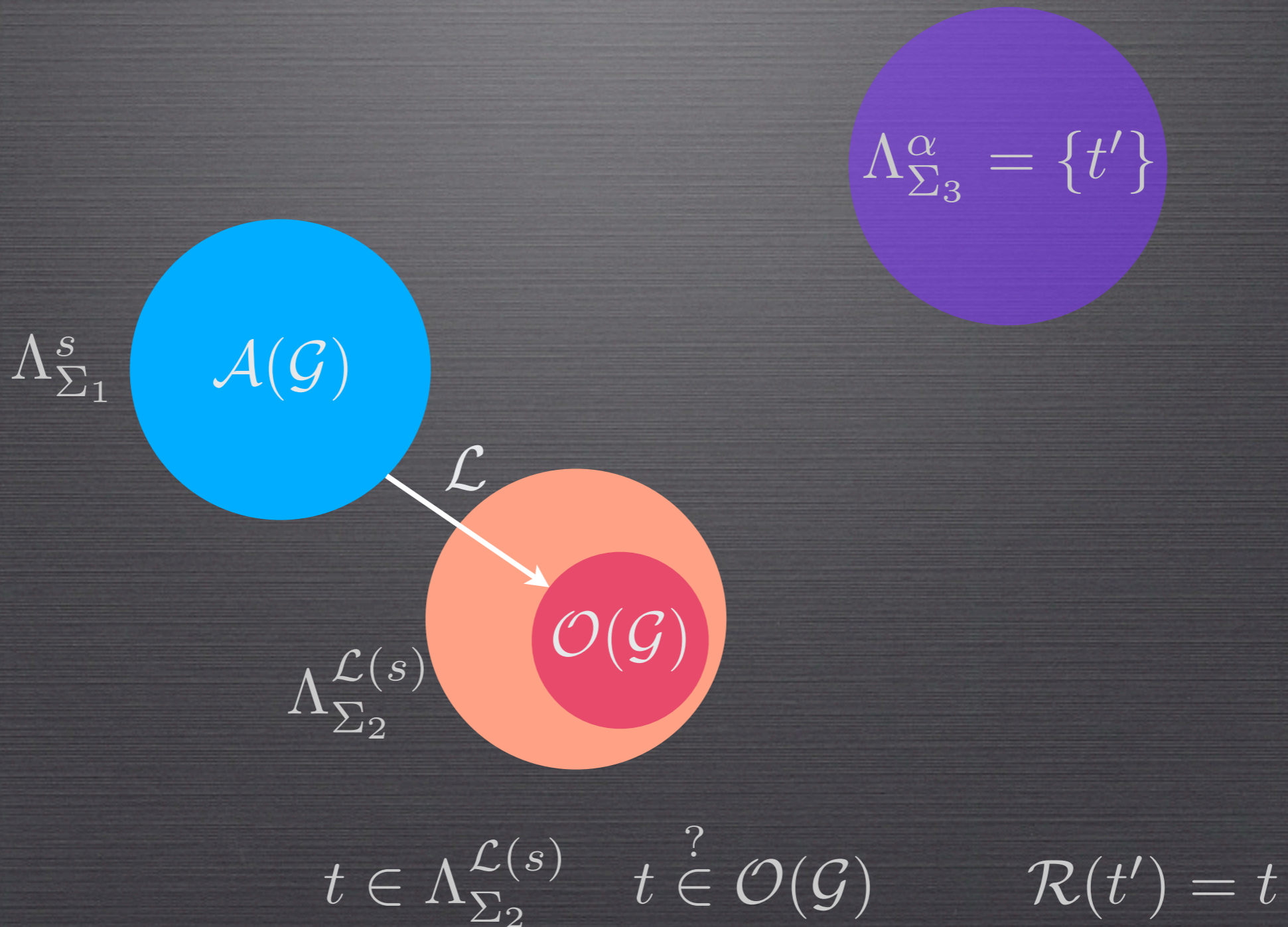
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MEMBERSHIP AND MELL



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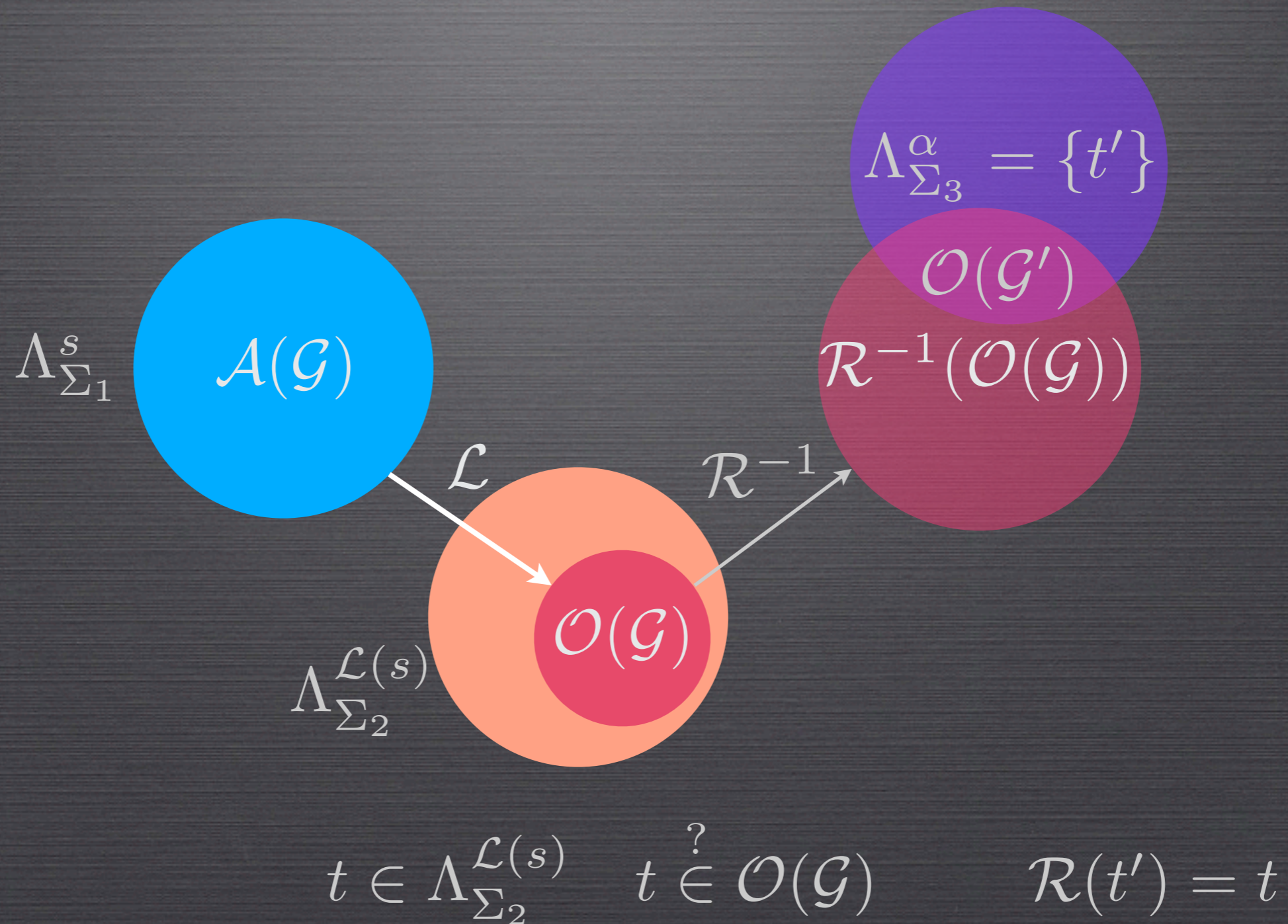
COHERENCE



MEMBERSHIP AND MELL

COHERENCE

CLOSURE UNDER INVERSE RELABELING



MEMBERSHIP AND MELL

EXERCISE:

REDUCE EMPTINESS OF AN ACG TO THE
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HINT 1: PROVE THAT SETS OF PURE TERMS OF A
GIVEN TYPE IS FINITE

MEMBERSHIP AND MELL

EXERCISE:

REDUCE EMPTINESS OF AN ACG TO THE
MEMBERSHIP PROBLEM

HINT 1: PROVE THAT SETS OF PURE TERMS OF A
GIVEN TYPE IS FINITE

HINT 2: PROVE THAT THE SETS OF PURE TERMS BUILT
WITH TYPES OF THE FORM

$$T ::= (\sigma \rightarrow \sigma) \mid (T \rightarrow T)$$

ARE NOT EMPTY

LEXICALIZED GRAMMARS

$\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$ IS LEXICALIZED IF FOR EVERY $c \in \Sigma_1$
 $\mathcal{L}(c)$ CONTAINS AT LEAST ONE OBJECT CONSTANT

LEXICALIZED GRAMMARS

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	Membership	Universal Membership
$\mathbf{G}(3, 1)$	NP-complete	NP-complete
$\mathbf{G}(3, n)$	NP-complete	NP-complete

COMPLEXITY OF

$G(2, n)$

	Membership	Universal Membership
$\mathbf{G}(2, 1)$	P	P
$\mathbf{G}(2, n) \ n > 1$	P	PSPACE-complete

CYK ALGORITHM FOR CFGS

LET $G = (N, T, R, s)$ BE A CFG

$a_1 \dots a_n$

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ITEMS (Y, i, j) MEANING $Y \xrightarrow{*} a_{i+1} \dots a_j$

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$$a_1 \dots a_n$$

ITEMS (Y, i, j) MEANING $Y \xrightarrow{*} a_{i+1} \dots a_j$

$$X \rightarrow w_1 X_1 \dots w_n X_n w_{n+1}$$

$$w_1 = a_i \dots a_j \quad (X_1, j+1, k)$$

\vdots

$$w_n = a_l \dots a_m \quad (X_n, m+1, p)$$

$$w_{n+1} = a_{p+1} \dots a_q$$

$$(X, i, q)$$

CYK ALGORITHM FOR $G(2, n)$

LET $\mathcal{G} = \langle \Sigma_1, \Sigma_2, \mathcal{L}, s \rangle$ BE IN $G(2, n)$ AND $t \in \Lambda_{\Sigma_2}^{\mathcal{L}(s)}$

COHERENCE: $\Sigma_3, \mathcal{R}, \alpha, t'$

ITEM: (β, γ) with $\beta \in \Sigma_1, \gamma \in \Sigma_2$ and $\mathcal{L}(\beta) = \mathcal{R}(\gamma)$

MEANING: there is $t \in \Lambda_{\Sigma_1}^{\beta}$ such that $\mathcal{R}^{-1}(\mathcal{L}(t)) \cap \Lambda_{\Sigma_3}^{\gamma} \neq \emptyset$

$$\frac{(\beta_1, \gamma_1) \cdots (\beta_n, \gamma_n) \quad \mathcal{R}^{-1}(\mathcal{L}(c)) \cap \Lambda_{\Sigma_3}^{\gamma_1 \rightarrow \cdots \rightarrow \gamma_n \rightarrow \gamma} \neq \emptyset}{(\beta, \gamma)}$$

NON LINEAR ACGS

ABSTRACT
SYNTACTIC
STRUCTURE

Σ_0

Σ_1

SYNTACTIC
FORM

ABSTRACT
SYNTACTIC
STRUCTURE

Σ_0

Σ_1

SYNTACTIC
FORM

Σ_2

SEMANTIC
FORM

ABSTRACT
SYNTACTIC
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Σ_0

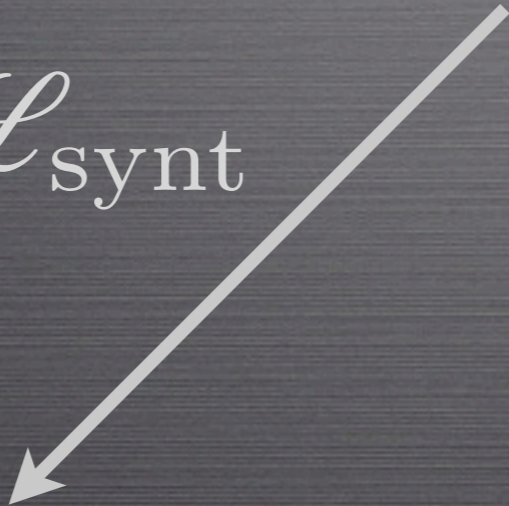
$\mathcal{L}_{\text{synt}}$

Σ_1

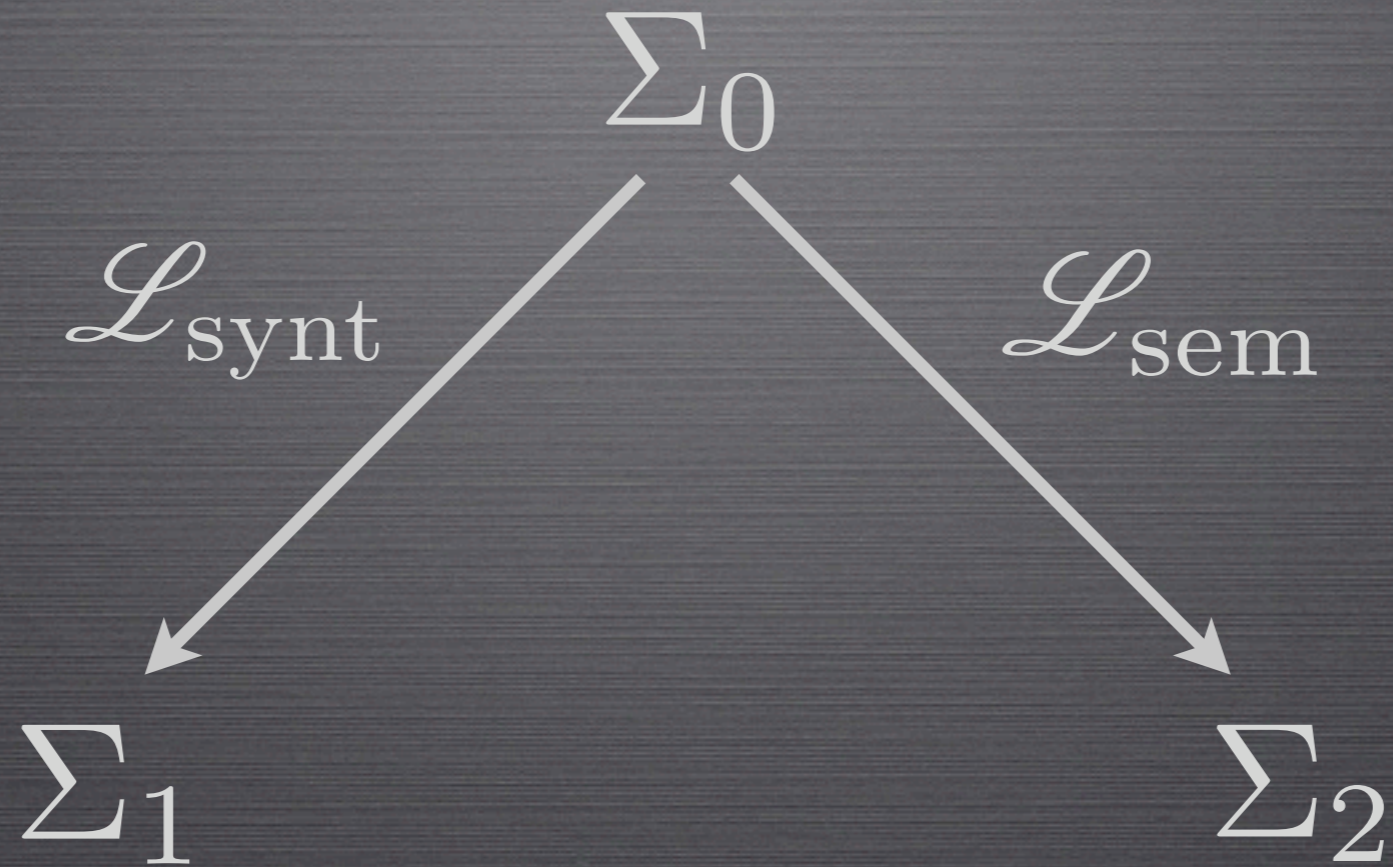
Σ_2

SYNTACTIC
FORM

SEMANTIC
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ABSTRACT
SYNTACTIC
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SYNTACTIC
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DECIDABILITY AND NON-LINEARITY

$G_{non-lin}(3, n)$: undecidable membership problem

$G_{non-lin}(2, n)$: decidable membership problem

$\exists x. \text{unicorn } x \wedge \text{find } j \ x$

DECIDABILITY OF $G_{non-linear}(2, n)$

- RECOGNIZABILITY FOR LAMBDA-TERMS
- SINGLETON SETS ARE RECOGNIZABLE
- RECOGNIZABLE SETS ARE CLOSED UNDER:
 - ▶ BOOLEAN OPERATIONS
 - ▶ INVERSE HOMOMORPHISMS (LEXICONS)