

# The Logic of Sense and Reference

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# Today

- Today we will have a look at two strategies to improve the granularity of meaning.
- Most of our time will be devoted to a **partialization** of type theory that I carried out in the 1980s (see Muskens 1995).
- This partialization lets more meanings become available, but, as far as problem of logical omniscience is concerned, it merely **alleviates** its consequences. It does not make the problem go away (I think the theory has other desirable properties, though).
- We will also consider an implementation in classical type logic of the **impossible possible worlds approach** to fine-grained meanings. This particular implementation is based on Muskens (1991).

# Overview

## 1 Partial Type Theory

- From a Functional to a Relational Logic
- From a Total to a Partial Logic
- Applications

## 2 Impossible Worlds in Classical Logic

# Irrelevancies in Classical Logic

- Classical logic supports many entailments that relevance logicians have deemed *irrelevant*.
- The following two statements, for example, co-entail classically:
  - (1) a. John is walking  
b. John is walking and Bill is talking or not talking
- As a consequence, the following are also predicted to co-entail:
  - (2) a. Mary thinks John is walking  
b. Mary thinks John is walking and Bill is talking or not talking
- But Mary may think John is walking without believing anything about Bill at all...

## Weeding out the Irrelevancies

- Barwise and Perry (1983) mention the irrelevancies just considered as one motivation for moving to **Situation Semantics**, a theory in which possible worlds are replaced by **situations**.
- A situation, intuitively, is a **part** of reality. A person's field of vision at some given time (a **scene**) can well be modeled as a situation, for example.
- Muskens (1995) argues that moving to a more fine-grained partial logic will weed out the irrelevancies just as well. Today, we will explain how this can be done.
- We will first move to a **relational** variant of type theory and then **partialize** it.

# Relational Type Theory: Types and Frames

## Definition

The set of types is the smallest set of strings such that:

- i. all basic types are types,
- ii. if  $\alpha_1, \dots, \alpha_n$  are types ( $n \geq 0$ ), then  $\langle \alpha_1 \dots \alpha_n \rangle$  is a type.

## Definition

A *frame* is a set of non-empty sets  $\{D_\alpha \mid \alpha \text{ is a type}\}$  such that

$$D_{\langle \alpha_1 \dots \alpha_n \rangle} \subseteq P(D_{\alpha_1} \times \dots \times D_{\alpha_n})$$

for all types  $\alpha_1, \dots, \alpha_n$ . A frame is *standard* if

$$D_{\langle \alpha_1 \dots \alpha_n \rangle} = P(D_{\alpha_1} \times \dots \times D_{\alpha_n})$$

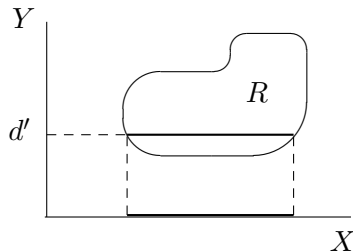
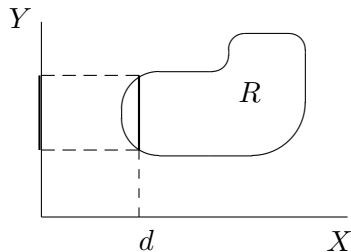
for all  $\alpha_1, \dots, \alpha_n$ .

# Comparison with Functional Types / The Domain $D_{\langle \rangle}$

- A functional type like  $e(e(st))$  (type of transitive verbs) will now become  $\langle ees \rangle$ , for example; the type  $(e(st))(st)$  will become  $\langle \langle es \rangle s \rangle$ .
- $\langle \alpha_1 \dots \alpha_n \rangle^\dagger = \alpha_1^\dagger(\alpha_2^\dagger(\dots(\alpha_n^\dagger t) \dots))$
- Note that in standard frames  

$$D_{\langle \rangle} = P(\{\langle \rangle\}) = P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\} = \{0, 1\}.$$
- So we get the truth and falsity domain as a limiting case.
- (This also holds for frames that underly a general model.)

# Relations as Functions



## Definition

Let  $R$  be an  $n + 1$ -ary relation. The *first slice function*  $F_R^1$  of  $R$  is given by:

$$F_R^1(d) = \{ \langle d_1, \dots, d_n \rangle \mid \langle d, d_1, \dots, d_n \rangle \in R \} .$$

# Terms

## Definition

Define, for each  $\alpha$ , the set of *terms* of that type as follows.

- i. Every constant or variable of any type is a term of that type;
- ii. If  $\varphi$  and  $\psi$  are terms of type  $\langle \rangle$  (*formulae*) then  $\neg\varphi$  and  $(\varphi \wedge \psi)$  are formulae;
- iii. If  $\varphi$  is a formula and  $x$  is a variable of any type, then  $\forall x \varphi$  is a formula;
- iv. If  $A$  is a term of type  $\langle \beta\alpha_1 \dots \alpha_n \rangle$  and  $B$  is a term of type  $\beta$ , then  $(AB)$  is a term of type  $\langle \alpha_1 \dots \alpha_n \rangle$ ;
- v. If  $A$  is a term of type  $\langle \alpha_1 \dots \alpha_n \rangle$  and  $x$  is a variable of type  $\beta$  then  $(\lambda x.A)$  is a term of type  $\langle \beta\alpha_1 \dots \alpha_n \rangle$ ;
- vi. If  $A$  and  $B$  are terms of the same type, then  $(A = B)$  is a formula.

# Truth Definition

## Definition

The *value*  $\|A\|^{M,a}$  of a term  $A$  on a model  $M$  under an assignment  $a$  is defined in the following way (To improve readability I shall sometimes write  $\|A\|$  for  $\|A\|^{M,a}$ ):

- i.  $\|c\| = I(c)$  if  $c$  is a constant;  
 $\|x\| = a(x)$  if  $x$  is a variable;
- ii.  $\|\neg\varphi\| = 1 - \|\varphi\|$ ;  
 $\|\varphi \wedge \psi\| = \|\varphi\| \cap \|\psi\|$ ;
- iii.  $\|\forall x_\alpha \varphi\|^{M,a} = \bigcap_{d \in D_\alpha} \|\varphi\|^{M,a[d/x]}$ ;
- iv.  $\|AB\| = F_{\|A\|}^1(\|B\|)$ ;
- v.  $\|\lambda x_\beta A\|^{M,a} =$  the  $R$  such that  $F_R^1(d) = \|A\|^{M,a[d/x]}$  for all  $d \in D_\beta$ ;
- vi.  $\|A = B\| = 1$  if  $\|A\| = \|B\|$ ;  
 $= 0$  if  $\|A\| \neq \|B\|$ .

# Entailment

## Definition

Let  $\Gamma$  and  $\Delta$  be sets of terms of some type  $\alpha = \langle \alpha_1 \dots \alpha_n \rangle$ .  $\Gamma$  is said to *s-entail*  $\Delta$ ,  $\Gamma \models_s \Delta$ , if

$$\bigcap_{A \in \Gamma} \|A\|^{M,a} \subseteq \bigcup_{B \in \Delta} \|B\|^{M,a}$$

for all standard models  $M$  and assignments  $a$  to  $M$ .

- The relational form of type theory is really just a **variant** of the more familiar functional form (no types like  $e \rightarrow e$ , though).
- We will use it as a stepping-stone to get to the **partial** theory of types.

# Partial Relations

## Definition

Let  $D_1, \dots, D_n$  be sets. An  $n$ -ary *partial relation*  $R$  on  $D_1, \dots, D_n$  is a tuple  $\langle R^+, R^- \rangle$  of relations  $R^+, R^- \subseteq D_1 \times \dots \times D_n$ .

If  $D$  is some set then the *partial power set* of  $D$ ,  $PP(D)$ , is  $P(D) \times P(D)$ , the set  $\{\langle R^+, R^- \rangle \mid R^+, R^- \subseteq D\}$  of all partial sets over  $D$ .

# Operations on Partial Relations

## Definition

Let  $R_1 = \langle R_1^+, R_1^- \rangle$  and  $R_2 = \langle R_2^+, R_2^- \rangle$  be partial relations. Define:

$$\begin{aligned}
 -R_1 &:= \langle R_1^-, R_1^+ \rangle \text{ (partial complementation)} \\
 R_1 \cap R_2 &:= \langle R_1^+ \cap R_2^+, R_1^- \cup R_2^- \rangle \text{ (partial intersection)} \\
 R_1 \cup R_2 &:= \langle R_1^+ \cup R_2^+, R_1^- \cap R_2^- \rangle \text{ (partial union)} \\
 R_1 \subseteq R_2 &\text{ iff } R_1^+ \subseteq R_2^+ \text{ and } R_2^- \subseteq R_1^- \text{ (partial inclusion)}
 \end{aligned}$$

Let  $A$  be some set of partial relations. Define:

$$\begin{aligned}
 \bigcap A &:= \langle \bigcap \{R^+ \mid R \in A\}, \bigcup \{R^- \mid R \in A\} \rangle \\
 \bigcup A &:= \langle \bigcup \{R^+ \mid R \in A\}, \bigcap \{R^- \mid R \in A\} \rangle.
 \end{aligned}$$

# Frames for Partial Type Theory

## Definition

A *frame* is a set of non-empty sets  $\{D_\alpha \mid \alpha \text{ is a type}\}$  such that

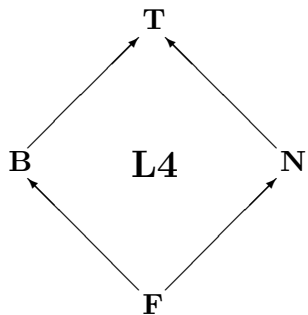
$$D_{\langle \alpha_1 \dots \alpha_n \rangle} \subseteq PP(D_{\alpha_1} \times \dots \times D_{\alpha_n}).$$

A frame is *standard* if  $D_{\langle \alpha_1 \dots \alpha_n \rangle} = PP(D_{\alpha_1} \times \dots \times D_{\alpha_n})$  for all  $\alpha_1, \dots, \alpha_n$ .

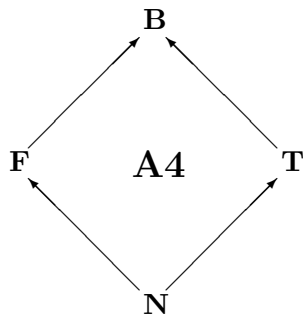
# The Domain $D_{\langle \rangle}$

- Note that in standard frames  $D_{\langle \rangle} = PP(\{\langle \rangle\}) = PP(\{\emptyset\}) = \{\langle R^+, R^- \rangle \mid R^+, R^- \subseteq \{\emptyset\}\} = \{\langle 1, 0 \rangle, \langle 0, 1 \rangle, \langle 0, 0 \rangle, \langle 1, 1 \rangle\}$ .
- So we now get **four** values in the truth value domain.
- (This also holds for frames that underly a general model.)
- We shall interpret  $\langle 1, 0 \rangle$  as ‘true and not false’ (**T**),  $\langle 0, 1 \rangle$  as ‘false and not true’ (**F**),  $\langle 0, 0 \rangle$  as ‘true nor false’ (**N**), and  $\langle 1, 1 \rangle$ , as ‘both true and false’ (**B**).
- We have arrived at the four values of **Belnap (1977)**.

# Belnap's Two Lattices



Logical lattice



Approximation lattice

In the logical lattice  $\subseteq$  gives the ordering. In the approximation lattice it is a natural dual.

# Four-valued Truth Tables

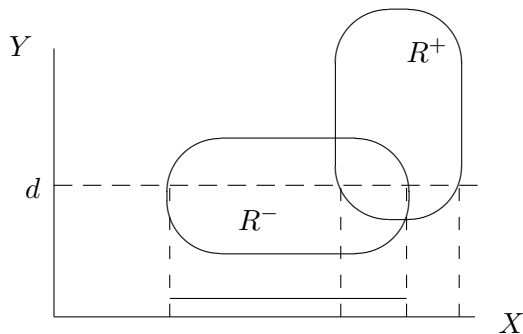
$\wedge$	<b>T</b>	<b>F</b>	<b>N</b>	<b>B</b>	$\vee$	<b>T</b>	<b>F</b>	<b>N</b>	<b>B</b>	$\neg$	
<b>T</b>	<b>T</b>	<b>F</b>	<b>N</b>	<b>B</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>F</b>
<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>T</b>	<b>F</b>	<b>N</b>	<b>B</b>	<b>F</b>	<b>T</b>
<b>N</b>	<b>N</b>	<b>F</b>	<b>N</b>	<b>F</b>	<b>N</b>	<b>T</b>	<b>N</b>	<b>N</b>	<b>T</b>	<b>N</b>	<b>N</b>
<b>B</b>	<b>B</b>	<b>F</b>	<b>F</b>	<b>B</b>	<b>B</b>	<b>T</b>	<b>B</b>	<b>T</b>	<b>B</b>	<b>B</b>	<b>B</b>

$\wedge$  corresponds to  $\cap$ ,  $\vee$  to  $\cup$ , and  $\neg$  to  $-$ .

# Irrelevancies Gotten Rid of

- $p$  does not entail  $p \wedge (q \vee \neg q)$ .  
(If  $p$  gets the value **T** and  $q$  gets the value **N**, then  $p \wedge (q \vee \neg q)$  gets the value **N**.)
- It is not the case that  $q \vee \neg q$  is always true.
- It is not the case that  $q \wedge \neg q$  is always false.
- $p \wedge \neg p \not\equiv q \vee \neg q$   
( $p$  could have the value **B** and  $q$  the value **N** or vice versa).

# Partial Relations as Functions



## Definition

Let  $R$  be an  $n + 1$ -ary partial relation. The *first slice function*  $F_R^1$  of  $R$  is defined by  $F_R^1(d) = \langle F_{R^+}^1(d), F_{R^-}^1(d) \rangle$ .

# Tarski Truth Definition

## Definition

The **value**  $\|A\|^{M,a}$  of a term  $A$  on a very general model  $M$  under an assignment  $a$  is defined in the following way:

- i.  $\|c\| = I(c)$  if  $c$  is a constant;  $\|x\| = a(x)$  if  $x$  is a variable;
- ii.  $\|\neg\varphi\| = -\|\varphi\|$ ;  $\|\varphi \wedge \psi\| = \|\varphi\| \cap \|\psi\|$ ;  
 $\|\#\| = \langle 1, 1 \rangle$ ;  $\|\star\| = \langle 0, 0 \rangle$ ;
- iii.  $\|\forall x_\alpha \varphi\|^{M,a} = \bigcap_{d \in D_\alpha} \|\varphi\|^{M,a[d/x]}$ ;
- iv.  $\|AB\| = F_{\|A\|}^1(\|B\|)$ ;
- v.  $\|\lambda x_\beta A\|^{M,a} =$  the  $R$  such that  $F_R^1(d) = \|A\|^{M,a[d/x]}$  for all  $d \in D_\beta$ ;
- vi.  $\|A = B\| = \langle 1, 0 \rangle$  if  $\|A\| = \|B\|$   
 $= \langle 0, 1 \rangle$  if  $\|A\| \neq \|B\|$ .

# Entailment

## Definition

Let  $\Gamma$  and  $\Delta$  be sets of terms of some type  $\alpha = \langle \alpha_1 \dots \alpha_n \rangle$ .  $\Gamma$  (*s*-)entails  $\Delta$ ,  $\Gamma \models \Delta$  ( $\Gamma \models_s \Delta$ ), if

$$\bigcap_{A \in \Gamma} \|A\|^{M,a} \subseteq \bigcup_{B \in \Delta} \|B\|^{M,a}$$

for all (standard) models  $M$  and assignments  $a$  to  $M$ .

Muskens (1995) gives an **Gentzen Calculus** that characterizes the generalised entailment relation  $\models$ .

(Here we have suppressed the definition of general models in the sense of Henkin (1950).)

## Finer Granularity (More Models, Less Entailments)

- Our partial logic gives finer granularity of meaning.
- But, because its syntax has hardly changed, still enables the usual Montague-like translation from language to logic.
- There no longer is co-entailment between sentences such as the following two:
  - (3) a. John is walking
  - b. John is walking and Bill is talking or not talking
- There can now be models in which (4a) is true but (4b) is false.
  - (4) a. Mary thinks John is walking
  - b. Mary thinks John is walking and Bill is talking or not talking
- The logical omniscience problem has not completely gone away, though. Sentences like the following perhaps illustrate it.
  - (5) a. Mary thinks John isn't walking and talking
  - b. Mary thinks John isn't walking or isn't talking

# Indices Become Possible Situations

- In the usual, total, set-up of formal semantics, the objects of type  $s$  behave like **possible worlds**.
- All sentences denote objects of type  $\langle s \rangle$  (or  $st$  in the functional approach). A world  $w$  (or rather the function  $\lambda p.pw$ ) can be viewed as assigning truth or falsity to every (denotation of a) sentence of the fragment: a complete description of what is the case.
- But in a partial (partial and paraconsistent) set-up, a function  $\lambda p.pw$  assigns the truth combinations **T**, **F**, **N**, **B** to any denotation of a sentence. This basically is a partial and paraconsistent description of what is the case.
- Indices have become **possible situations**.

## And Now For Something Completely Different

- The first part of today's lecture dealt with a **generalization** of classical type theory.
- Because of our partialisation we obtained more models and a richer structure on them.
- Irrelevancies can be avoided and worlds are replaced by situations.
- The remainder of today's lecture will be used for giving an **implementation of impossible worlds in classical type theory**.
- While the basic idea is different, there will be some obvious analogies with our treatment of Thomason's idea of having **propositions as primitives**.

# The Basic Idea

- Impossible worlds are worlds where the “logical” words do not have their standard meaning.
- This can be modeled by treating the “logical” words as **non**-logical constants, as we did in the implementation of Thomason’s idea.
- And by then stipulating that these constants must have the standard logical meaning of the word they are associated with at all **possible** worlds.
- For entailment only the **actual** world is relevant, which is of course possible; for meaning all possible and impossible worlds.

## Some Constants and their Types

non-logical constants	type
not	$(st)(st)$
and, or, if	$(st)((st)(st))$
every, a, some, no	$(e(st))((e(st))(st))$
is	$((e(st))(st))(e(st))$
hesperus, phosphorus, mary	$(e(st))(st)$
planet, man, woman, walk, talk	$e(st)$
believe, know	$(st)(e(st))$
$\Omega$	$st$
@	$s$
h, p, m	$e$

$\Omega$ —‘is a possible world’; @—the actual world

## Some Terms of Type $st$

- (6) (some woman)walk
- (7) (no man)talk
- (8) hesperus (is (a planet))
- (9) (if((a woman)walk))((no man)talk)
- (10) (if((a man)talk))((no woman)walk)
- (11) mary(believe((if((a woman)walk))((no man)talk)))
- (12) mary(believe((if((a man)talk))((no woman)walk)))

Haven't we seen these before? Well, they were terms of type  $p$  then...

# Logically Possible Possible Worlds

A1  $\forall i(\Omega i \rightarrow \forall p(\text{not } pi \leftrightarrow \neg pi))$

A2  $\forall i(\Omega i \rightarrow \forall pq(\text{and } pqi \leftrightarrow (pi \wedge qi)))$

A3  $\forall i(\Omega i \rightarrow \forall pq(\text{or } pqi \leftrightarrow (pi \vee qi)))$

A4  $\forall i(\Omega i \rightarrow \forall pq(\text{if } pqi \leftrightarrow (pi \rightarrow qi)))$

A5  $\forall i(\Omega i \rightarrow \forall P_1 P_2(\text{every } P_1 P_2 i \leftrightarrow \forall x(P_1 x i \rightarrow P_2 x i)))$

A6  $\forall i(\Omega i \rightarrow \forall P_1 P_2(\text{a } P_1 P_2 i \leftrightarrow \exists x(P_1 x i \wedge P_2 x i)))$

A7  $\forall i(\Omega i \rightarrow \forall P_1 P_2(\text{no } P_1 P_2 i \leftrightarrow \neg \exists x(P_1 x i \wedge P_2 x i)))$

A8  $\forall i(\Omega i \rightarrow \forall Q \forall x(\text{is } Q x i \leftrightarrow Q(\lambda y \lambda j . x = y) i))$

A9  $\forall i(\Omega i \rightarrow \forall P(\text{hesperus } P i \leftrightarrow P h i))$

$\forall i(\Omega i \rightarrow \forall P(\text{phosphorus } P i \leftrightarrow P p i))$

$\forall i(\Omega i \rightarrow \forall P(\text{mary } P i \leftrightarrow P m i))$

A10  $\Omega @$

# Hesperus is a Planet

Many sentences now get their **usual** denotation at the actual world. As an example, we show that  $\text{hesperus}(\text{is(a planet)})@$  is equivalent with  $\text{planet } h@$ .

$$\text{hesperus}(\text{is(a planet)})@ \quad (\text{is(a planet)})h@ \quad (\text{A9})$$

$$\text{a planet}(\lambda y \lambda j. h = y)@ \quad (\text{A8})$$

$$\exists x(\text{planet } x@ \wedge (\lambda y \lambda j. h = y)x@) \quad (\text{A6})$$

$$\exists x(\text{planet } x@ \wedge h = x) \quad (\beta)$$

$$\text{planet } h@ \quad (\text{predicate logic})$$

# Weak Entailment

## Definition

Let  $\varphi_1, \dots, \varphi_n, \psi$  be terms of type  $st$ . We say that  $\psi$  **follows from**  $\varphi_1, \dots, \varphi_n$  if A1–A10,  $\varphi_1@, \dots, \varphi_n@ \models \psi@$ . Terms  $\varphi$  and  $\psi$  of type  $st$  are called **equivalent** if  $\psi$  follows from  $\varphi$  and  $\varphi$  follows from  $\psi$ .

Essentially, this brings back standard logic to sentences not containing operators shifting the world of evaluation outside the set of possible possible worlds  $\Omega$ .

# Walk and Talk

The following are equivalent:

$$\begin{aligned} & (\text{if}((\text{a woman})\text{walk}))((\text{no man})\text{talk})@ \\ & ((\text{a woman})\text{walk})@ \rightarrow ((\text{no man})\text{talk})@ \end{aligned} \quad (\text{A4})$$

$$\exists x(\text{woman } x@ \wedge \text{walk } x@) \rightarrow ((\text{no man})\text{talk})@ \quad (\text{A6})$$

$$\exists x(\text{woman } x@ \wedge \text{walk } x@) \rightarrow \neg \exists x(\text{man } x@ \wedge \text{talk } x@) \quad (\text{A7})$$

$$\exists x(\text{man } x@ \wedge \text{talk } x@) \rightarrow \neg \exists x(\text{woman } x@ \wedge \text{walk } x@) \quad (\text{prop. logic})$$

$$\exists x(\text{man } x@ \wedge \text{talk } x@) \rightarrow ((\text{no woman})\text{walk})@ \quad (\text{A7})$$

$$((\text{a man})\text{talk})@ \rightarrow ((\text{no woman})\text{walk})@ \quad (\text{A6})$$

$$(\text{if}((\text{a man})\text{talk}))((\text{no woman})\text{walk})@ \quad (\text{A4})$$

So,  $(\text{if}((\text{a woman})\text{walk}))((\text{no man})\text{talk})$  and  $(\text{if}((\text{a man})\text{talk}))((\text{no woman})\text{walk})$  are equivalent, as they should be.

# Embedding into a Context of Propositional Attitude

- $\text{mary}(\text{believe}(\text{if}(\text{a woman walk})\text{no man talk}))@$
- $\text{believe}(\text{if}(\text{a woman walk})\text{no man talk})m@$
- No further reduction possible!
- Since  $\text{if}(\text{a woman walk})\text{no man talk}$  and  $\text{if}(\text{a man talk})\text{no woman walk}$  may take different values at impossible possible worlds, there are models in which Mary stands in the belief relation to one proposition but not the other.

# Conclusion

- If a fine-grained notion of meaning is wanted, we need to have more propositions than ordinary possible worlds semantics allows.
- In today's first approach classical type theory was partialized. This gives lots more propositions (which now become partial sets of situations).
- But while this approach provides us with interesting and useful structures, it does not provide us with a complete solution to the individuation problem.
- The impossible worlds approach provides us with a hyperfine-grained semantics.
- But are impossible worlds acceptable from a conceptual point of view?

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